SUMMATION OF DIFFERENCES

INVERSIONS AND DETERMINANTS

GROAT

AN INTRODUCTION

TO THE

SUMMATION OF DIFFERENCES OF A FUNCTION

AN ELEMENTARY EXPOSITION OF THE NATURE OF THE ALGEBRAIC PROCESSES REPLACED BY THE ABBREVIATIONS OF THE INFINITESIMAL CALCULUS

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PREFACE

The subject-matter for the following pages has its origin in various sources well known to the general mathematical reader; yet it is felt that the view of the differential and integral calculus presented in the sequel generally does not come to be recognized by the student sufficiently early in his course. The view referred to is suggested in a number of excellent works upon infinitesimal calculus, but the writer's aim is to treat the subject of this paper as a chapter in algebra, preparatory and supplementary to a course in differential and integral calculus, and not as a part of that course. The historical order of development has been followed in general outline, though the exercises are set in modern notation. The historical order of development of a science is usually (not always; cf. Hist. logarithms) the easiest to follow; moreover any other method withholds history and often fails to give a *sufficient* account of the science involved. This stands to reason for the actual development is likely to be the result of *necessity*. The reader is referred to several excellent articles and books for the history, which, after reading this paper, should be intelligible to the beginner.

The object is not merely to gain the historical view of the infinitesimal analysis but to prepare the student for the solution of problems in applied mathematics. The processes of differentiation and integration are acquired without much difficulty; but to *see the integral* with facility in a problem in analytic mechanics or physics requires clear notions as to sums and limits of sums. Such notions are of much more importance to the physicist and engineer than the more elaborate methods for in-

tegrating complicated forms: it is the desire to aid the student in forming these notions early, together with the writer's need of a suitable exercise book for use in his classes, that has been the reason for writing this paper.

The following is suggested as a course in fundamental principles and exercises: Elementary algebra, including progressions; convergence and divergence of series including the elementary test theorems; sums of squares and cubes of first n integers; undetermined coefficients and decomposition of fractions; exponential and logarithmic series and logarithms; elements of trigonometry; summation of series as in Chapter I, this book; derived functions; theory of equations, graphs and elementary limit theorems as stated in Arts. 7, 8, 9 and Theorem I, Art. 14, McMahon and Snyder's *Differential Calculus*; permutations, combinations, binomial theorem; determinants, system of linear equations, elimination, Sylvester's Method, discriminants; analytic geometry of plane and space; Chapter II, this book; differential calculus proper, that is, the rules and formulas of differentiation; Chapters III and IV; integral calculus followed by more complete courses.

For a simple demonstration of the logarithmic series the following is suggested: $\varepsilon^{y \log (1+x)} \equiv (1+x)^y$; expanding by the exponential and binomial theorems and equating the coefficients of y in the two expansions we obtain the logarithmic series. Of course the limitations of this proof should be noticed. The formulas for the sums of squares and cubes may be proved by induction. An early introduction to the factor and remainder theorems with their application in drawing graphs and locating roots of rational functions is advocated.

The kindness of Mr. H. H. Dalaker in reading proofs and verifying examples is acknowledged.

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SUMMATION OF DIFFERENCES

CHAPTER I

SUMMATION

1. The Symbol Σ. The series with which we shall have to deal are such that the nth term can be expressed as a function of n. Arithmetical and geometrical series are of this kind. Thus the nth term of the arithmetical series, $a+(a+d)+(a+2d)+\cdots$, is $(a+\overline{n-1}\,d)$; any term may be written from this by substituting the corresponding value of n. As an example the third term of the series is $(a+\overline{3-1}\,d)=(a+2d)$, as it should be. In the geometric series $a+ar+ar^2+\cdots$, the nth term is ar^{n-1}, and from this, for example, the fifth term is $ar^{5-1}=ar^4$, as it should be. Hence we see that the terms of an arithmetic progression are of the type $(a+\overline{x-1}\,d)$, and the terms of a geometric progression of the type ar^{x-1}, where x is to have the particular value corresponding to the number of the term in question.

2. The symbol $\sum$ is employed for the purpose of indicating that a sum of terms is to be taken. We define

$$\sum (a+\overline{x-1}\,d)$$

to mean the sum of terms of the type $(a+\overline{x-1}\,d)$,

$$\sum ar^{x-1}$$

to mean the sum of terms of the type ar^{x-1}, and in general

$$\sum \phi(x)$$

to mean the sum of terms of the type $\phi(x)$. Our definition, however, is not complete, since nothing in the notation indicates how many terms of a given type are to be included in the sum. Suppose, for example, we wish to indicate the sum of the first five terms of the series 3, 5, 7, 9, $\cdots$; we should write simply $\sum(3+2\overline{x-1})$, and have to explain that we are to sum the five values found by making x successively equal to 1, 2, 3, 4, 5, in $(3+2\overline{x-1})$. We complete the definition by attaching the *limits* of x in the following manner:

$$\sum_{x=1}^{x=5}(3+2\overline{x-1});$$

the limits designating the first and last values to be given to x. The limits may be omitted in any problem provided they are understood.

3. The student will have no difficulty in verifying the following illustrative examples:

$$\sum_3^7 2(x+1) = 2\cdot 4+2\cdot 5+2\cdot 6+2\cdot 7+2\cdot 8 = \frac{8+16}{2}\cdot 5 = 60.$$

$$\sum_3^7 2^{x-1} = 2^2+2^3+2^4+2^5+2^6 = \frac{2^2(2^5-1)}{2-1} = \frac{4\cdot 31}{1} = 124.$$

$$\sum_1^7 x = 1+2+\cdots+7 = \frac{1+7}{2}7 = 28.$$

$$\sum_1^{10} x^2 = 1^2+2^2+\cdots+10^2 = \frac{10(10+1)(20+1)}{6} = \frac{10\cdot 11\cdot 21}{6}$$
$$= 5\cdot 11\cdot 7 = 385.$$

$$\sum_1^7 x^3 \equiv \left\{\sum_1^7 x\right\}^2 \equiv \left\{7\frac{1+7}{2}\right\}^2 = 784.$$

$$\sum_3^5\left(x^2+\frac{1}{x}\right) \equiv \sum_3^5 x^2+\sum_3^5\frac{1}{x} = 3^2+4^2+5^2+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$
$$= 50+\frac{47}{60} = 50\tfrac{47}{60}.$$

$$\sum_{z=3}^{z=5} \log z = \log 60.$$

$$\sum_{3}^{6} (-1)^{z-1} \log z = \log \frac{5}{8}.$$

$$\sum_{n=5}^{10} (-1)^{n-1} \frac{1}{n} = -\sum_{5}^{10} (-1)^{n} \frac{1}{n} = \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}.$$

$$\sum_{m}^{n} \psi(x) \equiv \psi(m) + \psi(m+1) + \cdots + \psi(n-1) + \psi(n).$$

$$\sum_{z=0}^{3} \sin\left(\frac{z\pi}{2} + x\right) \cos\left(\frac{z\pi}{2} - x\right) = \frac{1}{2} \sum \sin z\pi + \frac{1}{2} \sum \sin 2x = 2 \sin 2x.$$

4. In the preceding paragraph we were concerned with finding the value of $\sum_{m}^{n} \phi(x)$; in the present and following article we shall solve the inverse problem, having given a series of the form $\phi(m) + \phi(m+1) + \cdots + \phi(n)$ to find the equivalent expression $\sum_{m}^{n} \phi(x)$. As an example take the series

$$\frac{3}{4 \cdot 5} - \frac{4}{5 \cdot 7} + \frac{5}{6 \cdot 9} - \frac{6}{7 \cdot 11} + \cdots - \frac{24}{25 \cdot 47}.$$

We observe at once that the signs of the even numbered terms are negative; this is effected in the summation by introducing the *sign factor* $(-1)^{z-1}$, so that when z is odd the sign is $+$. We notice further that each numerator is greater by 2 than the number of the term in which it stands, while the first factor of the denominator is 3 greater. The last factor of the denominator increases 2 units per term, and is 5 in the first term, hence it is $3+2n$ in the nth term; that is, it exceeds twice the number of the term by 3. The nth term, therefore, expressed as a function of n, may be written

$$(-1)^{n-1} \frac{(n+2)}{(n+3)(2n+3)},$$

and, all together, there are 22 terms.

$$\therefore \frac{3}{4 \cdot 5} - \frac{4}{5 \cdot 7} + \cdots - \frac{24}{25 \cdot 47} = \sum_{x=1}^{x=22} \frac{(-1)^{x-1}(x+2)}{(x+3)(2x+3)}.$$

5. Let the student verify the following equalities:

$$\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\cdots+\frac{1}{\underline{|10}}=\sum_{z=4}^{z=10}\frac{1}{\underline{|z}}.$$

$$\frac{3^4}{\underline{|4}}+\frac{3^5}{\underline{|5}}+\cdots+\frac{3^7}{\underline{|7}}=\sum_{n=5}^{8}\frac{3^{n-1}}{\underline{|n-1}}.$$

$$2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+7^{\frac{1}{7}}=\sum_{2}^{7}x^{\frac{1}{x}}.$$

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{x^5}{5}\equiv\sum_{z=1}^{5}(-1)^{z-1}\frac{x^z}{z}.$$

$$\frac{1}{1}-\frac{1}{1+x}+\frac{1}{1+2x}-\frac{1}{1+3x}\equiv\sum_{n=1}^{4}\frac{(-1)^{n-1}}{1+\overline{n-1}\,x}.$$

$$\frac{1}{x}+\frac{1}{1+x}+\frac{1}{2+x}+\cdots+\frac{1}{17+x}\equiv\sum_{z=0}^{17}\frac{1}{z+x}.$$

$$\frac{z^{16}+z^{\frac{1}{4}}}{\log m^4}+\frac{z^{25}-z^{\frac{1}{5}}}{\log m^5}+\frac{z^{36}+z^{\frac{1}{6}}}{\log m^6}\equiv\sum_{y=4}^{6}\frac{z^{y^2}+(-1)^y z^{\frac{1}{y}}}{y\log m}.$$

6. The following identities are obvious, being consequences of our definitions and the ordinary rules of algebra:

$$\sum_m^n\phi(x)\equiv\left\{\sum_m^r\phi(x)\right\}+\left\{\sum_{r+1}^s\phi(x)\right\}+\left\{\sum_{s+1}^t\phi(x)\right\}+\cdots+\left\{\sum_{\mu+1}^n\phi(x)\right\}. \qquad (a)$$

ILLUSTRATION. $\sum_1^{12}z^2=\sum_1^4 z^2+\sum_5^6 z^2+\sum_7^{10}z^2+\sum_{11}^{12}z^2.$

$$\left.\begin{array}{ll}\sum_a^b\phi(x)\equiv\sum_{a-p}^{b-p}\phi(y+p)\equiv\sum_{a-p}^{b-p}\phi(x+p), & \sum_m^n\phi(x+p)\equiv\sum_{m+p}^{n+p}\phi(x)\\ \sum_m^n\phi(x)\equiv\sum_{m+r}^{n+r}\phi(x-r), & \sum_a^b\phi(x-r)\equiv\sum_{a-r}^{b-r}\phi(x)\end{array}\right\}. \qquad (b)$$

ILLUSTRATIONS. $\sum_{3}^{5}\frac{1}{z^2}=\sum_{2}^{4}\frac{1}{(z+1)^2}$, $\sum_{5}^{9}\frac{1}{(x+2)^2}=\sum_{7}^{11}\frac{1}{x^2}$,

$\sum_{x=3}^{5} z^x=\sum_{3\frac{1}{2}}^{5\frac{1}{2}} z^{x-\frac{1}{2}}$, $\sum_{z=5}^{9} x^{z-2}=\sum_{3}^{7} x^z$.

$$\sum_{x=s}^{t}\left\{\phi(x)+\psi(x)+\cdots+f(x)\right\}\equiv\left\{\sum_{s}^{t}\phi(x)\right\}$$

$$+\left\{\sum_{s}^{t}\psi(x)\right\}+\cdots+\left\{\sum_{s}^{t}f(x)\right\}. \qquad (c)$$

SCHOLIUM. The braces may be omitted from the second members of (a) and (c), and when there are several summations of the same function for different sets of limits, as in (a), we may further abbreviate thus:

$$\sum_{m}^{n}\phi(x)\equiv\left\{\sum_{m}^{r}+\sum_{r+1}^{s}+\sum_{s+1}^{t}+\cdots+\sum_{\mu+1}^{n}\right\}\phi(x).$$

SCHOLIUM. We suppose in this article that none of the quantities involved are infinite.

7. Summation of Series. With the aid of the relations treated in the preceding article the summation of many series of the kind we are considering can be effected. This can always be done if the series is one whose nth term can be expressed as the **difference** between two expressions, one of which is the same function of $(n\pm p)$ that the other is of n, p being an integer. When this is not the case, other artifices must be employed, illustrations of which will be given, but the device alluded to above is the one of fundamental importance and to which most of our attention will be directed.

EXERCISES

1. Sum the series $u_1+u_2+u_3+\cdots+u_n+\cdots$, where u_n is capable of being expressed in the form $\phi(x)-\phi(x+p)]_{x=n}$.

SOLUTION. $$\sum_{x=1}^{x=n} u_x = \sum_1^n \left[\phi(x) - \phi(x+p)\right]$$

$$= \sum_1^n \phi(x) - \sum_1^n \phi(x+p)$$

$$= \left[\sum_1^n - \sum_{p+1}^{n+p}\right]\phi(x)$$

$$= \left[\sum_1^p + \sum_{p-1}^n - \sum_{p+1}^n - \sum_{n+1}^{n+p}\right]\phi(x)$$

$$= \sum_1^p \phi(x) - \sum_{n+1}^{n+p} \phi(x).$$

2. Find the sum of the $(n - m + 1)$ terms from the mth to the nth, both inclusive, in Ex. 1.

SOLUTION. $$\sum_m^n u_x = \sum_m^n \phi(x) - \sum_m^n \phi(x+p)$$

$$= \left[\sum_{m-p}^{n-p} - \sum_m^n\right]\phi(x+p)$$

$$= \left[\sum_{m-p}^{m-1} + \sum_m^{n-p} - \sum_m^{n-p} - \sum_{n-p+1}^n\right]\phi(x+p)$$

$$= \left[\sum_m^{m+p-1} - \sum_{n+1}^{n+p}\right]\phi(x)$$

$$= \sum_m^{m+p-1} \phi(x) - \sum_{n+1}^{n+p} \phi(x).$$

SCHOLIUM. There are fewer terms in $\sum_m^{m+p-1} - \sum_{n+1}^{n+p}$ than in $\sum_m^n$ only when $2p < (n - m + 1)$. The most important case is when $p = 1$, this giving two terms in the result, no matter how many in the given sum. The sum to infinity is found by passing to the limit when $n \doteq \infty$;

thus, $$\sum_1^\infty u_x = \sum_m^{m+p-1} \phi(x) - \underset{n \doteq \infty}{\mathrm{Lt}} \sum_{n+1}^{n+p} \phi(x).$$

3. Find the sum of n terms of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ and also the *sum of the series.*

SOLUTION. The nth term is $\frac{1}{n(n+1)}$. But by decomposition of fractions $\frac{1}{n(n+1)}\equiv\frac{1}{n}-\frac{1}{n+1}$.

$$\therefore \sum_1^n \frac{1}{n(n+1)}=\sum_1^n \frac{1}{x}-\sum_1^n \frac{1}{x+1}$$

$$=\frac{1}{1}+\sum_2^n \frac{1}{x}-\sum_1^{n-1}\frac{1}{x+1}-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}.$$

Therefore, $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=1.$$

4. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\cdots=?$

SOLUTION. $$\frac{1}{z(z+2)}\equiv\frac{1}{2}\left(\frac{1}{z}-\frac{1}{z+2}\right).$$

$$\therefore \sum_{n=1}^{n=r}\frac{1}{n(n+2)}=\frac{1}{2}\left[\sum_1^r\frac{1}{n}-\sum_1^r\frac{1}{n+2}\right]$$

$$=\frac{1}{2}\left[\sum_1^2+\sum_3^r-\sum_3^r-\sum_{r+1}^{r+2}\right]\frac{1}{n}$$

$$=\frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{r+1}-\frac{1}{r+2}\right].$$

Hence the sum of r terms from the beginning is $\frac{3}{4}-\frac{1}{2}\left[\frac{1}{r+1}+\frac{1}{r+2}\right]$.

Passing to the limit when $r\doteq\infty$, we find that $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\cdots=\frac{3}{4}$.

5. $\sum_1^n \frac{1}{x(x+3)(x+6)}=?$

SOLUTION. $$\frac{1}{x(x+6)}\equiv\frac{1}{6}\left[\frac{1}{x}-\frac{1}{x+6}\right].$$

$$\therefore \frac{1}{x(x+3)(x+6)}\equiv\frac{1}{6}\left[\frac{1}{x(x+3)}-\frac{1}{(x+3)(x+6)}\right].$$

Whence $\sum_1^n \frac{1}{x(x+3)(x+6)} = \frac{1}{6}\left[\sum_1^n \frac{1}{x(x+3)} - \sum_1^n \frac{1}{(x+3)(x+6)}\right]$

$$= \frac{1}{6}\left[\sum_1^n - \sum_4^{n+3}\right]\frac{1}{x(x+3)}$$

$$= \frac{1}{6}\left[\sum_1^3 + \sum_4^n - \sum_4^n - \sum_{n+1}^{n+3}\right]\frac{1}{x(x+3)}$$

$$= \frac{1}{6}\left[\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} - \frac{1}{(n+1)(n+4)} - \frac{1}{(n+2)(n+5)} - \frac{1}{(n+3)(n+6)}\right].$$

The sum to infinity is $\frac{73}{1080}$.

6. $\sum_1^\infty \frac{x+3}{x(x+1)(x+2)} = \frac{5}{4}$.

SOLUTION. $\frac{n+3}{n(n+1)(n+2)} \equiv \frac{1}{2}\left[\frac{n+3}{n(n+1)} - \frac{n+3}{(n+1)(n+2)}\right]$.

$$\therefore \sum_1^n \frac{x+3}{x(x+1)(x+2)} = \frac{1}{2}\left[\sum_1^n \frac{x+3}{x(x+1)} - \sum_1^n \frac{x+3}{(x+1)(x+2)}\right]$$

$$= \frac{1}{2}\left[2 + \sum_2^n \frac{x+3}{x(x+1)} - \sum_2^n \frac{x+2}{x(x+1)} - \frac{n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[2 + \sum_2^n \frac{1}{x(x+1)} - \frac{n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[2 + \sum_2^n \left(\frac{1}{x} - \frac{1}{x+1}\right) - \frac{n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[2 + \left(\sum_2^n - \sum_3^{n+1}\right)\frac{1}{x} - \frac{n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[2 + \left(\frac{1}{2} - \frac{1}{n+1}\right) - \frac{n+3}{(n+1)(n+2)}\right]$$

$$= \frac{5}{4} - \frac{1}{2}\left[\frac{1}{n+1} + \frac{n+3}{(n+1)(n+2)}\right]$$

is the sum of n terms. Passing to the limit, we derive the result required.

7. Find the sum of n terms of $\dfrac{2}{3\cdot 5}-\dfrac{3}{5\cdot 7}+\text{etc.}$

SOLUTION. $$\frac{n+1}{(2n+1)(2n+3)}\equiv\frac{1}{2}\left(\frac{n+1}{2n+1}-\frac{n+1}{2n+3}\right).$$

$$\therefore \sum_1^n\frac{(-1)^{y-1}(y+1)}{(2y+1)(2y+3)}=\frac{1}{2}\left[\sum_1^n(-1)^{y-1}\frac{y+1}{2y+1}+\sum_1^n(-1)^y\frac{y+1}{2y+3}\right]$$

$$=\frac{1}{2}\left[\frac{2}{3}+\sum_2^n(-1)^{y-1}\frac{y+1}{2y+1}+\sum_2^n(-1)^{y-1}\frac{y}{2y+1}+(-1)^n\frac{n+1}{2n+3}\right]$$

$$=\frac{1}{2}\left[\frac{2}{3}+\sum_2^n(-1)^{y+1}+(-1)^n\frac{n+1}{2n+3}\right]$$

$$=\frac{1}{2}\left[\frac{2}{3}-\frac{1}{2}+\frac{(-1)^{n-1}}{2}+(-1)^n\frac{n+1}{2n+3}\right].$$

8. $$\frac{1}{\underline{|2}}+\frac{2}{\underline{|3}}+\frac{3}{\underline{|4}}+\cdots+\frac{n}{\underline{|n+1}}=?$$

SUGGESTION. $$\frac{x}{\underline{|x+1}}\equiv\frac{1}{\underline{|x}}-\frac{1}{\underline{|x+1}}.$$

9. $$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots=?$$

10. Sum to infinity the series

$$\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\cdots.$$

SUGGESTION. $$\frac{2x+1}{x^2(x+1)^2}\equiv\frac{1}{x^2}-\frac{1}{(x+1)^2}.$$

11. $$\sum_1^\infty\frac{1}{(n+2)\underline{|n}}=\frac{1}{2}.$$

SUGGESTION. $$\frac{1}{\underline{|n+1}}-\frac{1}{\underline{|n+2}}\equiv\frac{1}{(n+2)\underline{|n}}.$$

12. Find the sum of any r consecutive terms, and also the sum to infinity of

$$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots.$$

13. $\frac{2}{1\cdot 3}\cdot\frac{1}{3}+\frac{3}{3\cdot 5}\cdot\frac{1}{3^2}+\frac{4}{5\cdot 7}\cdot\frac{1}{3^3}+\cdots=?$

SUGGESTION. The nth term should reduce to

$$\frac{1}{4}\left(\frac{1}{2n-1}\,\frac{1}{3^{n-1}}-\frac{1}{2n+1}\,\frac{1}{3^n}\right).$$

Ans. $\begin{cases}\text{Sum of } n \text{ terms is } \frac{1}{4}\left[1-\frac{1}{(2n+1)3^n}\right].\\ \text{Sum to infinity is } \frac{1}{4}.\end{cases}$

14. Sum to n terms the series

$$\cos x+\cos(x+y)+\cos(x+2y)+\cdots$$

SOLUTION. $2\cos x\sin y\equiv\sin(x+y)-\sin(x-y).$

$$\therefore\ 2\cos(x+\overline{n-1}\,y)\sin\frac{y}{2}\equiv\sin\left(x+\overline{2n-1}\,\frac{y}{2}\right)-\sin\left(x+\overline{2n-3}\,\frac{y}{2}\right);$$

whence

$$2\sin\frac{y}{2}\sum_{n=1}^{r}\cos(x+\overline{n-1}\,y)=\sum_{1}^{r}\sin\left(x+\overline{2n-1}\,\frac{y}{2}\right)-\sum_{0}^{r-1}\sin\left(x+\overline{2n-1}\,\frac{y}{2}\right)$$

$$=\sin\left(x+\overline{2r-1}\,\frac{y}{2}\right)-\sin\left(x-\frac{y}{2}\right).$$

Hence the sum of n terms is

$$\frac{\sin\left(x+\overline{2n-1}\,\frac{y}{2}\right)-\sin\left(x-\frac{y}{2}\right)}{2\sin\frac{y}{2}}.$$

15. $\sum_{s}^{t}\sin(a+zb)=?$

16. $\frac{1}{1\cdot 3}+\frac{2}{1\cdot 3\cdot 5}+\frac{3}{1\cdot 3\cdot 5\cdot 7}+\cdots+\frac{n}{1\cdot 3\cdot 5\cdots\overline{2n+1}}.$

Before proceeding with the solution we give a definition quite similar to the definition of $\sum_{a}^{b}\phi(x)$, namely:

$$\prod_{x=m}^{x=n}\psi(x)\equiv\psi(m)\psi(m+1)\psi(m+2)\cdots\psi(n).$$

SOLUTION.

$$\frac{n}{1\cdot 3\cdot 5\cdots\overline{2n+1}}\equiv\frac{1}{2}\left[\frac{1}{1\cdot 3\cdot 5\cdots\overline{2n-1}}-\frac{1}{1\cdot 3\cdot 5\cdots\overline{2n+1}}\right].$$

$$\therefore\ \sum_{z=1}^{z=n}\frac{z}{\prod\limits_{x=1}^{x=z}\overline{2x+1}}=\frac{1}{2}\left[\sum_{z=1}^{n}\frac{1}{\prod\limits_{x=1}^{z}\overline{2x-1}}-\sum_{z=1}^{n}\frac{1}{\prod\limits_{x=1}^{z}\overline{2x+1}}\right]$$

$$=\frac{1}{2}\left[\sum_{z=1}^{n}\frac{1}{\prod\limits_{x=0}^{z-1}\overline{2x+1}}-\sum_{z=1}^{n}\frac{1}{\prod\limits_{x=1}^{z}\overline{2x+1}}\right]$$

$$=\frac{1}{2}\left[\sum_{z=0}^{n-1}-\sum_{z=1}^{n}\right]\frac{1}{\prod\limits_{x=0}^{z}\overline{2x+1}}$$

$$=\frac{1}{2}\left[1-\frac{1}{\prod\limits_{x=0}^{x=n}\overline{2x+1}}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{1\cdot 3\cdot 5\cdots\overline{2n+1}}\right].$$

SCHOLIUM. Solve $\sum\frac{n}{\underline{\left|n+1\right.}}$ by this method.

8. SCHOLIUM. In solving these examples the sums of the $(m-n+1)$ terms from the nth to the mth, both inclusive, should be determined also. Additional examples will be found in C. Smith's *Treatise on Algebra.*

CHAPTER II

LIMITS

Before beginning the study of this chapter it is necessary to know the definition of a limit, the meaning of the symbol $\doteq$, and the elementary limit theorems mentioned in the preface.

9. Theorem. *The limit toward which the values of the fractions* $\frac{\tan x}{x}$, $\frac{x}{\tan x}$ *approach indefinitely near, when the value of x is taken indefinitely near zero, is unity;* or in notation,

$$\left.\frac{\tan x}{x}\right]_{x\doteq 0} \doteq 1; \quad \left.\frac{x}{\tan x}\right]_{x\doteq 0} \doteq 1.$$

Proof. We know that $\tan x > x > \sin x$.*

$$\therefore \qquad 1 > \frac{x}{\tan x} > \cos x.$$

If x† be taken sufficiently near to zero, $\cos x$ can be made to differ from unity by a quantity as small as you please; therefore, with greater reason, $\frac{x}{\tan x}$, which is less than unity but greater than $\cos x$, can, by taking x small enough, be made to differ as little as you please from unity. Hence, the theorem $\left.\frac{x}{\tan x}\right]_{x\doteq 0} \doteq 1$.

Let the student prove the second part, *i.e.* $\left.\frac{\tan x}{x}\right]_{x\doteq 0} \doteq 1$.

It should be noticed that we do not say $\frac{x}{\tan x}$, or $\frac{\tan x}{x}$, ever does really *equal* unity, but that unity is the *limit* toward

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* The student should demonstrate this inequality, referring to some trigonometry or geometry if he has forgotten the proof. See Levett and Davison's *Plane Trigonometry*, Vol. I, p. 78, Cor.

† Here we use x to mean the *value* of x, and hereafter shall frequently do the same with reference also to other symbols.

which either fraction approaches indefinitely near when x is taken indefinitely small.

COR. $$\underset{x \doteq 0}{\text{Lt}} \frac{\tan x}{x} = \underset{x \doteq 0}{\text{Lt}} \frac{x}{\tan x} = 1.$$

10. Theorem. $$\left.\frac{\sin x}{x}\right]_{x \doteq 0} \doteq 1 \text{ and } \left.\frac{x}{\sin x}\right]_{x \doteq 0} \doteq 1.$$

Let the student give the proof.

COR. $$\underset{x \doteq 0}{\text{Lt}} \frac{\sin x}{x} = \underset{x \doteq 0}{\text{Lt}} \frac{x}{\sin x} = 1.$$

EXERCISES

1. $\left.\dfrac{\tan ax}{x}\right]_{x \doteq 0} \doteq ?$

SOLUTION. $$\left.\frac{\tan ax}{x}\right]_{x=0} \equiv \frac{0}{0}, \text{ which is indeterminate.}$$

But $$\frac{\tan ax}{x} \equiv a\frac{\tan ax}{ax} \equiv a\frac{\tan z}{z}, \text{ if } z \equiv ax.$$

$$\therefore \underset{x \doteq 0}{\text{Lt}} \frac{\tan ax}{x} \equiv \underset{x \doteq 0}{\text{Lt}} a\frac{\tan ax}{ax} = a \underset{x \doteq 0}{\text{Lt}} \frac{\tan ax}{ax} = a \underset{z \doteq 0}{\text{Lt}} \frac{\tan z}{z}.$$

$$\therefore \underset{x \doteq 0}{\text{Lt}} \frac{\tan ax}{x} = a.$$

2. Find $\underset{x \doteq 0}{\text{Lt}} \dfrac{x}{\sin mx}$. *Ans.* $\dfrac{1}{m}$.

3. Find $\underset{x \doteq 0}{\text{Lt}} \dfrac{ax}{\tan x}$. *Ans.* a.

4. $\left.(b-x)\cot(b-x)\right]_{x \doteq b} \doteq ?$

SOLUTION. $$\left.(b-x)\cot(b-x)\right]_{x=b} = 0 \times \infty \text{ is indeterminate.}$$

But $$(b-x)\cot(b-x) \equiv \frac{(b-x)}{\tan(b-x)} \text{ and } \underset{x \doteq b}{\text{Lt.}} \frac{(b-x)}{\tan(b-x)} = 1.$$

$$\therefore \underset{x \doteq b}{\text{Lt}} (b-x)\cot(b-x) = 1.$$

5. Find $\underset{x \doteq b}{\text{Lt}} (c-x)\cot(b-x),\ c \neq b.$

SUGGESTION. $(c-x)\cot(b-x) \equiv \dfrac{(c-x)}{(b-x)} \cdot \dfrac{(b-x)}{\tan(b-x)}.$

Ans. ∞.

6. Find $\underset{x \doteq 2}{\text{Lt}} (2-x)\tan\dfrac{\pi}{x}.$

$$\left.(2-x)\tan\frac{\pi}{x}\right]_{x=2} \text{ is indeterminate.}$$

But $(2-x)\tan\dfrac{\pi}{x} \equiv \dfrac{2-x}{\cot\dfrac{\pi}{x}} \equiv \dfrac{2-x}{\tan\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)} \equiv \dfrac{2-x}{\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)} \cdot \dfrac{\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)}{\tan\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)}.$

Now $\underset{x \doteq 2}{\text{Lt}} \dfrac{\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)}{\tan\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)} = 1$; hence our answer is

$\left.\dfrac{2-x}{\dfrac{\pi}{2}-\dfrac{\pi}{x}}\right]_{x \doteq 2}$, that is $\underset{x \doteq 2}{\text{Lt}} \dfrac{2-x}{\pi\dfrac{x-2}{2x}} = \underset{x \doteq 2}{\text{Lt}} \dfrac{2x}{\pi}(-1) = -\dfrac{4}{\pi}.$

7. Find the limit, $x \doteq 0$, of $x\tan\left(\dfrac{\pi}{2}-ax\right).$ *Ans.* $\dfrac{1}{a}.$

8. Same as Ex. 5, when $x \doteq c$.

11. An Application of Principles. Draw a circle of radius a. Divide the radius into n (say 8) equal parts, and also the arc of the quadrant whose origin is the extremity of the divided radius. At each point of division of the radius erect a perpendicular, and from each point of division of the quadrant draw a radius. Draw a curve through the points of intersection of the perpendiculars and corresponding radii, beginning at the origin. This curve is the *quadratrix*. Take the origin of the arc for the origin, O, of rectangular coördinates, the divided radius being the axis of X. Let two points, P and P_1, move uniformly, one in the arc of the quadrant, the other along the radius, so that both start from the origin at the same instant and arrive, P at the extremity of the arc of the quadrant, and P_1 at the center,

C, of the circle, at the same instant. It is clear that the perpendicular from P_1 intersects the radius from P in a point of the curve, and that the vertical intercept is the ordinate y, of that point; that $OP_1 = x$ and $\frac{OP}{a} = \alpha$, the measuring arc to the angle PCO. Also $\frac{\alpha}{x} = \frac{\pi}{2a}$. Let the student show that

$$y = (a - x)\tan\frac{\pi x}{2a}$$

is the equation of the curve.

Now $y = \dfrac{(a-x)}{\cot\dfrac{\pi x}{2a}} \equiv \dfrac{(a-x)}{\tan\left(\dfrac{\pi}{2} - \dfrac{\pi x}{2a}\right)} \equiv \dfrac{2a}{\pi}\,\dfrac{\dfrac{\pi}{2}\left(\dfrac{a-x}{a}\right)}{\tan\dfrac{\pi}{2}\left(\dfrac{a-x}{a}\right)}$,

and $\dfrac{\dfrac{\pi}{2}\left(\dfrac{a-x}{a}\right)}{\tan\dfrac{\pi}{2}\left(\dfrac{a-x}{a}\right)}$ becomes indefinitely near to unity when x is taken indefinitely near to a; therefore

$$y\Big]_{x \doteq a} \doteq \frac{2a}{\pi}.$$

Hence, if y_0 is the ordinate for $x = a$,

$$\pi = \frac{2a}{y_0} = \frac{\text{Diameter of circle}}{\text{Maximum ordinate}}.$$

We have then a mechanical means of determining an approximate value of π, by simply dividing the diameter of the circle by the maximum ordinate as found by construction. An approximate value of π (one of the earliest determinations) was found in some such manner by **Dinostratus**, about 400 B.C. The curve is frequently called the *Quadratrix of Dinostratus*, but was invented about half a century before the time of his *quadrature* by **Hippias**, supposed of Elis. The curve is also employed in the trisection (*transcendental*) of the arc, and, indeed, may be used to find any fraction of an arc. An irregular draughting curve is of great assistance in the construction.

Check the construction by plotting two or three points of the quadratrix from the equation

$$y = (a - x) \tan \frac{\pi x}{2a}.$$

12. Archimedes (born 287 B.C.) of Syracuse, the greatest mathematician of antiquity, found that the circumference of a circle was "greater than three times the diameter by a fraction less than $\frac{10}{70}$ and greater than $\frac{10}{71}$ of the diameter." His method was to inscribe and circumscribe regular polygons of 96 sides within and about a circle, and then from actual measurement to compare the perimeters with the diameter. Thus if P_i and P_c are respectively the perimeters of the inscribed and circumscribed polygons, and R the radius of the circle, he found

$$\frac{P_c}{2R} = 3\tfrac{1}{7} > \pi > \frac{P_i}{2R} = 3\tfrac{10}{71}.$$

Let the student explain the double inequality and also make an estimate of $\frac{P_c}{2R}$ and $\frac{P_i}{2R}$ by measurement, after inscribing and circumscribing regular polygons of 24 sides in and about a circle of 5 or 6 inches radius. The following inequalities were established in this way:

$$\frac{P_c}{2R} = \frac{22.8}{7.2} = 3.17 > \pi > \frac{P_i}{2R} = \frac{22.56}{7.2} = 3.13.$$

13. Theorem. $\left.\left(1 + \frac{1}{x}\right)^x\right]_{x \doteq \pm\infty} \doteq e$, where e is a number which lies between 2 and 3.

PROOF. Let m be a positive integer. Then

$$\left(1 + \frac{1}{m}\right)^m \equiv 1 + 1 + \frac{\left(1 - \frac{1}{m}\right)}{\lfloor 2} + \frac{\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{\lfloor 3}$$

$$+ \cdots + \frac{\left(1 - \frac{1}{m}\right) \cdots \left(1 - \frac{m-1}{m}\right)}{\lfloor m}$$

by the binomial theorem, and

$$\left(1+\frac{1}{m+1}\right)^{m+1} \equiv 1+1+\frac{\left(1-\frac{1}{m+1}\right)}{\underline{|2}}+\frac{\left(1-\frac{1}{m+1}\right)\left(1-\frac{2}{m+1}\right)}{\underline{|3}}$$
$$+\cdots+\frac{\left(1-\frac{1}{m+1}\right)\cdots\left(1-\frac{m}{m+1}\right)}{\underline{|m+1}}$$

by changing m to $m+1$. Whence it is plain that

$$\left(1+\frac{1}{m+1}\right)^{m+1} > \left(1+\frac{1}{m}\right)^{m},$$

since, after two terms, each term of $\left(1+\frac{1}{m+1}\right)^{m+1}$ is greater than the corresponding term of $\left(1+\frac{1}{m}\right)^{m}$, and, moreover, there is one more term in $\left(1+\frac{1}{m+1}\right)^{m+1}$. Hence, *when x is increased by unity, x being any positive integer,* $\left(1+\frac{1}{x}\right)^{x}$ *increases.*

Again, it is clear that $\left(1+\frac{1}{m}\right)^{m}$ can never be greater than the series

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots+\frac{1}{\underline{|m}}=1+\sum_{1}^{m}\frac{1}{\underline{|z}}.$$

This last series is less than

$$1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{m-1}}=1+\sum_{1}^{m}\frac{1}{2^{y-1}}=3-\frac{1}{2^{m-1}}.$$

Whence $$\left(1+\frac{1}{m}\right)^{m}<3-\frac{1}{2^{m-1}},$$

m being a positive integer. If we take m to be a very large positive integer, we see that $\left(1+\frac{1}{m}\right)^{m}<3$ however large m may be, since it is always less than a quantity which can never be greater than 3. Therefore, since $\left(1+\frac{1}{z}\right)^{z}$ increases continually when z does, z being a positive integer, but at the same time is always less than 3, and since $\left.\left(1+\frac{1}{z}\right)^{z}\right]_{z=1}=2$, we see

that $\left(1+\frac{1}{z}\right)^z$ must approach some limiting value which lies between 2 and 3. Call this limit e, and the theorem is proved for a positive integer.

To prove the theorem for a positive *fractional* value of x. Let θ be a positive fraction, and m and n two consecutive integers, such that

$$m = n + 1, \text{ and } m > \theta > n.$$

Then
$$\left(1+\frac{1}{m}\right) < \left(1+\frac{1}{\theta}\right) < \left(1+\frac{1}{n}\right),$$

and
$$\left(1+\frac{1}{m}\right)^\theta < \left(1+\frac{1}{\theta}\right)^\theta < \left(1+\frac{1}{n}\right)^\theta;$$

that is,
$$\left(1+\frac{1}{m}\right)^{m-\alpha} < \left(1+\frac{1}{\theta}\right)^\theta < \left(1+\frac{1}{n}\right)^{n+\beta},$$

α and β being positive proper fractions; hence

$$\frac{\left(1+\frac{1}{m}\right)^m}{\left(1+\frac{1}{m}\right)} < \left(1+\frac{1}{\theta}\right)^\theta < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right).$$

If θ is made to increase without limit, m and n consequently do the same,

$$\left(1+\frac{1}{m}\right) \doteq 1, \quad \left(1+\frac{1}{n}\right) \doteq 1, \quad \left(1+\frac{1}{m}\right)^m \doteq e, \text{ and } \left(1+\frac{1}{n}\right)^n \doteq e;$$

therefore $\left(1+\frac{1}{\theta}\right)^\theta \doteq e$, θ being a positive fraction, and the theorem is proved for positive fractional values of x.

To prove the theorem for *negative* values of x. Let x be any positive number, and put $y = -x$; hence y is numerically equal to x, but negative. We have

$$\left(1+\frac{1}{y}\right)^y \equiv \left(1-\frac{1}{x}\right)^{-x} \equiv \left(\frac{x-1}{x}\right)^{-x} \equiv \left(\frac{x}{x-1}\right)^x \equiv \left(1+\frac{1}{x-1}\right)^x$$

i.e.
$$\left(1+\frac{1}{y}\right)^y \equiv \left(1+\frac{1}{x-1}\right)^{x-1}\left(1+\frac{1}{x-1}\right).$$

Let $x \doteq \infty$; then

$$y \doteq -\infty, \quad \left(1 + \frac{1}{x-1}\right)^{x-1} \doteq e, \text{ and } \left(1 + \frac{1}{x-1}\right) \doteq 1.$$

Hence the limit, when y approaches negative infinity, of $\left(1 + \frac{1}{y}\right)^y$ is e.

Therefore $\left.\left(1 + \frac{1}{x}\right)^x\right]_{x \doteq \pm\infty} \doteq e$ whether x be positive, negative, integral, or fractional. Q.E.D.

Scholium. For other proofs of this theorem see Todhunter's *Differential Calculus*.

14. The Value of *e*. The student may be interested to know the value of e. By definition, $e = \underset{x \doteq \infty}{\text{Lt}} \left(1 + \frac{1}{x}\right)^x$, and this in whatever way x is made to approach infinity, whether positive, negative, integral, or fractional. Then let m be a positive integer; whence

$$\left(1 + \frac{1}{m}\right)^m \equiv 1 + 1 + \frac{\left(1 - \frac{1}{m}\right)}{\lfloor 2} + \cdots + \frac{\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{n-2}{m}\right)}{\lfloor n-1}$$
$$+ \cdots + \frac{\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{m-1}{m}\right)}{\lfloor m}.$$

Let the sum of the first n terms of this series be denoted by S_n, and the remainder after n terms by R_n; that is,

$$R_n \equiv \frac{\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right)}{\lfloor n} + \cdots + \frac{\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{m-1}{m}\right)}{\lfloor m},$$

the remaining $(m + 1 - n)$ terms of S_{m+1}.

Hence
$$R_n < \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1} + \cdots + \frac{1}{\lfloor m}$$
$$< \frac{1}{\lfloor n}\left\{1 + \frac{1}{n} + \cdots + \frac{1}{n^{m-n}}\right\}.$$

$$\therefore \qquad R_n < \left[\frac{1}{\underline{|n}} \frac{1}{1 - \frac{1}{n}} = \frac{1}{n-1} \frac{1}{\underline{|n-1}} \right],$$

and
$$\left(1 + \frac{1}{m}\right)^m = S_n + R_n.$$

It is clear that if n be any *finite* number $\underset{m \doteq \infty}{\text{Lt}} S_n = 1 + 1 + \frac{1}{\underline{|2}} + \cdots + \frac{1}{\underline{|n-1}}$, which call u_n; that is to say, $S_n = u_n - \rho$, where ρ is a positive quantity that can be made as small as you please by taking m sufficiently large, n being finite. We have then

$$S_n < S_{m+1} = S_n + R_n < S_n + \frac{1}{n-1} \frac{1}{\underline{|n-1}},$$

or
$$u_n - \rho < S_{m+1} < u_n - \rho + \frac{1}{n-1} \frac{1}{\underline{|n-1}}.$$

Passing to the limit when $m \doteq \infty$, we have

$$u_n < e < u_n + \frac{1}{n-1} \frac{1}{\underline{|n-1}}.$$

If n be taken sufficiently large, $\frac{1}{n-1} \frac{1}{\underline{|n-1}}$ can be made as small as you please, and therefore, if n is large enough, u_n will differ from e by a quantity smaller than $\frac{1}{n-1} \frac{1}{\underline{|n-1}}$; that is, smaller than a quantity which is as small as you please. Hence

$$e = u_\infty = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \text{etc.}$$

We may now find the value of e to as high a degree of approximation as desired. For example,

$$e = 1 + \sum_1^\infty \frac{1}{\underline{|z}} \equiv 1 + \sum_1^8 \frac{1}{\underline{|z}} + \sum_9^\infty \frac{1}{\underline{|z}},$$

and
$$\sum_9^\infty \frac{1}{\underline{|z}} < \left[\frac{1}{8} \frac{1}{\underline{|8}} < 0.000004 \right].$$

Hence the sum of the first nine terms of $1 + \sum_1^\infty \frac{1}{\underline{|s}}$ differs from e by less than 0.000004, and consequently will give the first six figures of e provided the sixth decimal figure of the sum is less than 6. For clearness the calculation is arranged as follows:

$$1 + \sum_1^{10} \frac{1}{\underline{|s}} < e < 1 + \sum_1^{8} \frac{1}{\underline{|s}} + \frac{1}{8}\frac{1}{\underline{|8}}.$$

$2.0000000 =$	$1 + 1 \div \underline{\|1}$	$= 2.000000$
$.5000000 =$	$1 \div \underline{\|2}$	$= .500000$
$.1666666 <$	$1 \div \underline{\|3}$	$< .166667$
$.0416666 <$	$1 \div \underline{\|4}$	$< .041667$
$.0083333 <$	$1 \div \underline{\|5}$	$< .008334$
$.0013888 <$	$1 \div \underline{\|6}$	$< .001389$
$.0001984 <$	$1 \div \underline{\|7}$	$< .000199$
$.0000248 <$	$1 \div \underline{\|8}$	$< .000025$
$.0000027 <$	$1 \div \underline{\|9}$; $1 \div 8\underline{\|8}$	$< .000004$
$.0000002 <$	$1 \div \underline{\|10}$;	2.718285
2.7182814		

Hence e lies between 2.7182814 and 2.718285, and consequently 2.71828 must be the first six figures of e. As a matter of fact 2.718281 are the first seven figures; but to prove this would require further investigation. By taking in more terms of e, and carrying each term sufficiently far out, we may, by reasoning analogous to the above, find the value of e as nearly as desired.

It will transpire that e is the base of the natural system of logarithms, an absolutely incommensurable number. It is called the **Napierian Base** in honor of Napier, the inventor of logarithms, and its value to thirty decimal places is

$$2.71828\ 18284\ 59045\ 23536\ 02874\ 71353$$

15. Theorem. $\left.(1+z)^{\frac{1}{z}}\right]_{z \doteq 0} \doteq e.$

PROOF. Put $\frac{1}{x} \equiv z$, then $\left(1+\frac{1}{x}\right)^x \equiv (1+z)^{\frac{1}{z}}$.

$$\therefore \qquad \underset{x \doteq \infty}{\mathrm{Lt}} \left(1+\frac{1}{x}\right)^x = \underset{z \doteq 0}{\mathrm{Lt}} (1+z)^{\frac{1}{z}} = e. \qquad \text{Q.E.D.}$$

16. Theorem. $\left.\frac{a^v - 1}{v}\right]_{v \doteq 0} \doteq \log_e a.$

Put $a^v \equiv z + 1$. Then if $v \doteq 0$, $z \doteq 0$. We have

$$\log_e (1+z)^{\frac{1}{z}} \equiv \frac{1}{z} \log_e (1+z) \equiv \frac{\log_e a^v}{a^v - 1}.$$

$$\therefore \qquad \underset{v \doteq 0}{\mathrm{Lt}} \frac{a^v - 1}{v} = \underset{z \doteq 0}{\mathrm{Lt}} \frac{\log_e a}{\log_e (1+z)^{\frac{1}{z}}} = \log_e a. \qquad \text{Q.E.D.}$$

COR. $\underset{v \doteq 0}{\mathrm{Lt}} \frac{v}{a^v - 1} = \log_a e.$

17. Summary. We have demonstrated the following important theorems:

$$\underset{x \doteq 0}{\mathrm{Lt}} \frac{\tan x}{x} = \underset{x \doteq 0}{\mathrm{Lt}} \frac{x}{\tan x} = 1;$$

$$\underset{x \doteq 0}{\mathrm{Lt}} \frac{\sin x}{x} = \underset{x \doteq 0}{\mathrm{Lt}} \frac{x}{\sin x} = 1;$$

$$\underset{x \doteq \infty}{\mathrm{Lt}} \left(1+\frac{1}{x}\right)^x = \underset{x \doteq 0}{\mathrm{Lt}} (1+x)^{\frac{1}{x}} = e;$$

$$\underset{x \doteq 0}{\mathrm{Lt}} \frac{a^x - 1}{x} = \log_e a;$$

$$\underset{x \doteq 0}{\mathrm{Lt}} \frac{x}{a^x - 1} = \log_a e;$$

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots = 2.71828+.$$

18. Application to Analytic Geometry. *Problem.* To find the slope of the tangent through a given point of a curve.

Let $y = f(x)$ be the equation of the curve and $P_1 \equiv (x_1, y_1)$ the given point. Take another point,* $P_2 \equiv (x_2, y_2)$, on the curve. Then $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope of the secant line through P_1, P_2. By taking P_2 sufficiently near to P_1, the secant P_1P_2 can be made to lie as near the tangent through P_1 as you please, and the slope of the secant can be made to differ as little as you please from the slope of the tangent; that is, if $P_2 \doteq P_1$,† then $y_2 \doteq y_1$, $x_2 \doteq x_1$, and we have

$$\mathop{\text{Lt}}_{\substack{x_2 \doteq x_1 \\ y_2 \doteq y_1}} \frac{y_2 - y_1}{x_2 - x_1} = \text{slope of tangent.}$$

Let the symbol $\Delta y \equiv y_2 - y_1$, and $\Delta x \equiv x_2 - x_1$; ‡

then
$$x_2 = x_1 + \Delta x, \quad y_2 = y_1 + \Delta y,$$

$$y_1 = f(x_1), \quad y_1 + \Delta y = f(x_1 + \Delta x),$$

and
$$\mathop{\text{Lt}}_{x_2 \doteq x_1} \frac{y_2 - y_1}{x_2 - x_1} = \mathop{\text{Lt}}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \mathop{\text{Lt}}_{\Delta x \doteq 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}.$$

Therefore the slope of the tangent through the point (x, y) of the curve $y = f(x)$ is

$$\mathop{\text{Lt}}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \mathop{\text{Lt}}_{h \doteq 0} \frac{f(x + h) - f(x)}{h}.$$

Let the student draw the figure and review the proof.

EXERCISES

1. Find the equation of the tangent to the curve $y = x^2$ through the point whose abscissa is 2. Thus, $x_1 = 2$, $y_1 = 4$.

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* Barrow's method for drawing a tangent.

† Here the symbol $\doteq$ means *literally* "approaches indefinitely near to."

‡ It is to be noticed that these definitions of Δy and Δx imply that Δy is a function of Δx vanishing with the latter, y being also a function of x.

SOLUTION. $\dfrac{\Delta y}{\Delta x} = \dfrac{(2+\Delta x)^2 - 2^2}{\Delta x} \equiv \dfrac{4\,\Delta x + \Delta x^2}{\Delta x} = 4 + \Delta x.$

$$\therefore \qquad \underset{\Delta x \doteq 0}{\text{Lt.}} \frac{\Delta y}{\Delta x} = \text{slope of tangent} = 4.$$

Ans. $(y-4) = 4(x-2)$.

2. Same as Ex. 1 for curve $y = x^3$. *Ans.* $y - 12x + 16 = 0$.

3. Find the tangent to the curve $y = \dfrac{1}{x}$ at the point (x_1, y_1).

SOLUTION. $\dfrac{f(x_1+h) - f(x_1)}{h} \equiv \dfrac{\dfrac{1}{x_1+h} - \dfrac{1}{x_1}}{h} \equiv \dfrac{-1}{x_1(x_1+h)}.$

Hence the slope of tangent is $-\dfrac{1}{x_1^2}$, and the equation is

$$(y - y_1) = -\frac{1}{x_1^2}(x - x_1).$$

4. Find the slope of the tangent to $y = \dfrac{1}{x^2}$ at the point (x, y).

Ans. $-\dfrac{2}{x^3}$.

5. Find the slope of the curve $y = \dfrac{1}{x^{\frac{1}{2}}}$.

SOLUTION. $\underset{\Delta x \doteq 0}{\text{Lt}} \dfrac{\Delta y}{\Delta x} = \underset{h \doteq 0}{\text{Lt}} \dfrac{\dfrac{1}{(x+h)^{\frac{1}{2}}} - \dfrac{1}{x^{\frac{1}{2}}}}{h} = \underset{h \doteq 0}{\text{Lt}} \dfrac{x^{\frac{1}{2}} - (x+h)^{\frac{1}{2}}}{hx^{\frac{1}{2}}(x+h)^{\frac{1}{2}}}$

$$= \underset{h \doteq 0}{\text{Lt}} \frac{-1}{x^{\frac{1}{2}}(x+h)^{\frac{1}{2}}[x^{\frac{1}{2}} + (x+h)^{\frac{1}{2}}]}.$$

Ans. $-\frac{1}{2}x^{-\frac{3}{2}}$.

6. Find the tangent to the curve $y = \dfrac{1}{x^{\frac{3}{2}}}$ at the point (x_1, y_1).

Ans. $y - y_1 = -\frac{3}{2}x_1^{-\frac{5}{2}}(x - x_1)$.

7. Find $\underset{h \doteq 0}{\text{Lt}} \dfrac{f(x+h) - f(x)}{h}$ for the following:

$$y = mx^{\frac{3}{2}}, \qquad y = c\sqrt{x}, \qquad y = \sqrt{x+m}, \qquad y = (x+b)^3.$$

8. Find the slope of $y = \sin x$.

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x+h) - \sin x}{h} \equiv \frac{2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{h}.$$

SUGGESTION. Use a theorem of the summary at end of Chapter II.

Ans. $\underset{\Delta x \doteq 0}{\text{Lt}} \dfrac{\Delta y}{\Delta x} = \cos x$.

9. $y = \log_e x$, $\qquad \underset{\Delta x \doteq 0}{\text{Lt}} \dfrac{\Delta y}{\Delta x} = \dfrac{1}{x}$.

10. $y = a^x$, $\qquad \underset{\Delta x \doteq 0}{\text{Lt}} \dfrac{\Delta y}{\Delta x} = a^x \log_e a$.

19. Notation. In the preceding problems our object was to find the slope of a tangent, which was found by performing the operation indicated by $\underset{\Delta x \doteq 0}{\text{Lt}} \dfrac{\Delta y}{\Delta x}$, or by $\left.\dfrac{f(x+h)-f(x)}{h}\right]_{h \doteq 0}$. But this may be done whether we consider $y = f(x)$ to be a curve or not; and the result is, in general, another function of x, derived by this process and called the "*first derived function*," "*first differential coefficient*," or simply the "*derivative*" or "*derivate*" of the function. Thus the first derived function of $f(x)$ is

$$\underset{h \doteq 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h},$$

and the first derived function of y considered as a function of x is

$$\underset{\Delta x \doteq 0}{\text{Lt}} \frac{\Delta y}{\Delta x},*$$

these being the expressions which fully indicate the operations to be performed upon $f(x)$ and y. We will also indicate this by the symbol $\dfrac{d}{dx}$ written before any function whose derivative with respect to x is to be found. Thus

$$\frac{d}{dx}(y), \text{ or } \frac{dy}{dx},\dagger \text{ means } \underset{\Delta x \doteq 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$$

and
$$\frac{d}{dx}f(x) \equiv \underset{h \doteq 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}.$$

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* Δy is by definition above the increment of y due to the increment Δx of x, y being also a function of x.

† The notation dy and dx for the *infinitesimal* increments of y and x was introduced by Leibnitz (*Leibniz*).

Another notation frequently employed is explained by either of the definitions

$$f'(x) \equiv \frac{d}{dx} f(x), \quad \phi'(x) \equiv \underset{h \doteq 0}{\text{Lt}} \frac{\phi(x+h) - \phi(x)}{h}.$$

The slope of the curve $y = \psi(x)$ may be expressed by any one of

$$\frac{d}{dx}(y), \quad \frac{dy}{dx}, \quad \underset{h \doteq 0}{\text{Lt}} \frac{\psi(x+h) - \psi(x)}{h}, \quad \underset{\Delta x \doteq 0}{\text{Lt}} \frac{\Delta y}{\Delta x}, \quad \psi'(x).$$

EXERCISES

1. $\frac{d}{dx}(x+4)^3 = 3(x+4)^2$; $\frac{d}{dx} e^x = e^x$.

2. $y = \sin ax$. $\frac{dy}{dx} = a \cos ax$.

3. $f(x) = x^n$, n being a positive integer. $f'(x) = nx^{n-1}$.

4. $\frac{d}{dx} \cos ax = -a \sin ax$; $\frac{d}{dx} \tan x = \sec^2 x$.

5. $\phi(x) = \frac{x+1}{x-1}$. $\phi'(x) = -\frac{2}{(x-1)^2}$.

6. $y = \log_e az$. $\psi'(z) = \frac{1}{z}$.

7. $\frac{d}{dz} m^z = ?$ $\frac{d}{dm} m^z = ?$

8. If $y = \log \phi(x)$, prove $\frac{dy}{dx} = \frac{\phi'(x)}{\phi(x)}$.

SOLUTION.

$$\frac{\Delta y}{\Delta x} = \frac{1}{h} \log \frac{\phi(x+h)}{\phi(x)}$$

$$= \frac{1}{h} \log \left[1 + \frac{\phi(x+h) - \phi(x)}{h} \cdot \frac{h}{\phi(x)} \right]$$

$$= \frac{1}{\phi(x)} \log \left[1 + \{\phi'(x) + \epsilon\} \frac{h}{\phi(x)} \right]^{\frac{\phi(x)}{h}} \qquad \epsilon]_{h \doteq 0} \doteq 0$$

$$= \frac{1}{\phi(x)} \log \left[(1+z)^{\frac{1}{z}} \right]^{\phi'(x)} + \frac{1}{\phi(x)} \log \left[(1+z)^{\frac{1}{z}} \right]^{\epsilon} \qquad z]_{h \doteq 0} \doteq 0.$$

$$\therefore \frac{dy}{dx} = \frac{\phi'(x)}{\phi(x)}.$$

9. Solve the first fifteen or more examples at page 50, Todhunter's *Differential Calculus*, by Fermat's method, indicated by

$$\underset{h \doteq 0}{\mathrm{Lt}} \frac{f(x+h)-f(x)}{h}.*$$

20. The Differential Calculus. We have seen how, by applying the formula $\frac{f(x+h)-f(x)}{h}$, and, after certain algebraic reductions, passing to the limit of the fraction, we have been able to find the first differential coefficient of $f(x)$. This process is called **differentiation**; and when a function has been thus operated upon, it is said to be "*differentiated*." Now as there are but six fundamental algebraic operations and only a few transcendental functions in common use, we ought to be able to devise simple rules and corresponding formulæ for writing out the differential coefficient of any such given form. For example, in our last lesson we found the derivative of x^n, n being a positive integer, to be nx^{n-1}. It can be shown that this result holds for all values of n; consequently we have the rule for differentiating any power of a variable with respect to that variable: "*Take the product of the exponent and the variable with its exponent diminished by unity.*" Thus: $\frac{d}{dx}x^7 = 7x^6$, $\frac{d}{dx}x^{-2} = -2x^{-3}$, $\frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}}$, etc., generally.

These rules and the formulæ expressing them in mathematical symbols constitute the **differential calculus.**† Hence, by means of a few simple rules we are able to tell what the

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* Strictly speaking, Fermat's method is one for determining the maxima and minima of functions of a single variable; but as it is so closely related to the fundamental formula, $\frac{f(x+h)-f(x)}{h}$, that Lagrange and other eminent mathematicians would have given him credit for the invention of the differential calculus, it seems quite proper to associate his name with the *general* formula of differentiation.

† Invented by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibnitz (1646–1716) independently. While Newton certainly had priority, yet the notation of Leibnitz was as certainly superior; and thus it is that the honors are quite evenly divided.

result of applying the formula $\underset{h \doteq 0}{\mathrm{Lt}} \frac{f(x+h)-f(x)}{h}$ is, without carrying out the indicated operations, which become in many cases exceedingly involved and difficult of execution. Herein lies the advantage of the differential calculus.

21. The student should learn the rules and formulæ immediately; and how to apply them. Each formula should be demonstrated *independently* of any other rule of the calculus, this being facilitated, in the case of the trigonometric, logarithmic, and inverse functions, by the summary of Art. 17: illustrated in the exercises of this chapter.

CHAPTER III

AREAS AND LIMITS OF SUMS

22. We proceed to examine a method for finding the area enclosed by a given curve, the axis of x, and any two ordinates; and thence a method for finding the limit of a sum of n terms of a certain form, when $n \doteq \infty$.

Let $y=f(x)$ be the equation of the curve, $f(x)$ being an increasing function in the first quadrant. Suppose we require the area enclosed by the curve, the ordinates $f(a)$, $f(b)$, and the x axis. Suppose $b>a$. Then the base of the area is $(b-a)$ in length. Divide the base into n equal parts and erect ordinates at the points of division. There will be $(n+1)$ of these ordinates, $y_1, y_2, \cdots y_{n+1}$, and their common distance apart will be $\dfrac{b-a}{n}$, which call $\Delta x \equiv h$. Further, we can calculate the lengths of these ordinates from the equations $y_1=f(a)$, $y_2=f(a+h), \cdots y_n=f(a+\overline{n-1}\,h)=f(b-h)$, $y_{n+1}=f(b)$.

If from the points of intersection of these ordinates with the curve, parallels to x be drawn in the positive direction to the next ordinate in each case, we shall form a system of n rectangles,

$$y_1h,\ y_2h,\ \cdots y_nh,$$

whose area,

$$y_1h+y_2h+\cdots+y_nh,$$

is less than the required area; each rectangle lying on the positive side of its generating ordinate. If the parallels are drawn in the negative direction, the area, $y_2h+y_3h+\cdots+y_{n+1}h$, will be greater than the required area, and each of the n rectangles will lie on the negative side of its generating ordinate.

If $f(x)$ be a decreasing function, the inequalities will be inverted. If $f(x)$ be part of the time increasing and part of

the time decreasing, or *vice versa*, the inequalities may or may not be inverted. In general we shall have inequalities; thus

$$\sum_{x=1}^{n} y_x h \lesssim A \lesssim \sum_{2}^{n+1} y_x h;$$

A being the area sought. We will call that system of rectangles lying to the right of their generating ordinates the system R; the other will be L. Then

$$R \equiv \sum_{1}^{n} y_x h, \quad L \equiv \sum_{2}^{n+1} y_x h.$$

Now $$L - R = \sum_{2}^{n+1} y_x h - \sum_{1}^{n} y_x h = (y_{n+1} - y_1) h;$$

that is, *the difference between the area of the L system and the area of the R system is the same as the difference between the areas of the last L rectangle and first R rectangle;* a very important result.

Let the student draw the figure and interpret the result graphically.

23. By taking h small enough, and consequently n large enough, $(y_{n+1} - y_1)h \equiv L - R$ can be made as small as you please. Therefore A, which lies *between* L and R, must differ from either by less than $|(y_{n+1} - y_1)h|$. Therefore

$$\underset{h \doteq 0}{\mathrm{Lt}} \sum_{1}^{n} y_x h = \underset{h \doteq 0}{\mathrm{Lt}} \sum_{2}^{n+1} y_x h = A.$$

Whenever one of these limits can be found, the result is the *exact* area required.

SCHOLIUM. Since h is a common factor of the terms of $\sum$, we may write,

$$A = \underset{h \doteq 0}{\mathrm{Lt}}\, h \sum_{1}^{n} y_x = \underset{n \doteq \infty}{\mathrm{Lt}}\, h \sum_{2}^{n+1} y_x.$$

24. The inequalities and reasoning above may be expressed in another way, employing the equations $y_1 = f(a)$, $y_2 = f(a + h)$, $\cdots y_n = f(b - h)$, $y_{n+1} = f(b)$, $h \equiv \Delta x$, and a slightly modified notation, thus:

$$f(a)h+f(a+h)h+\cdots+f(b-2h)h+f(b-h)h\equiv\sum_{x=a}^{b-\Delta x}f(x)\Delta x\equiv R, \quad (a)$$

$$f(a+h)h+f(a+2h)h+\cdots+f(b-h)h+f(b)h\equiv\sum_{a+\Delta x}^{x=b}f(x)\Delta x\equiv L. \quad (b)$$

This notation means that x is increased by Δx from term to term, from a to $(b-\Delta x)$ in the first summation, and from $(a+\Delta x)$ to b in the second. This implies that $(b-a)$ is an exact multiple of Δx; but that follows since $\dfrac{b-a}{\Delta x}=n$, a positive integer by definition. It is not necessary, however, that this be so, nor indeed that the Δx's be all equal, for inequalities of the kind considered can always be devised by having

$$\sum_{z=1}^{n}\Delta x_z<(b-a)<\sum_{z=1}^{n+1}\Delta x_z.$$

The question of incommensurability will not be more closely touched upon here.

Hence from (a) and (b),

$$\sum_{a}^{b-\Delta x}f(x)\Delta x\lesseqgtr A\lesseqgtr\sum_{a+\Delta x}^{b}f(x)\Delta x.$$

But $\sum_{a+\Delta x}^{b}-\sum_{a}^{b-\Delta x}\equiv[f(b)-f(a)]\Delta x$ vanishes with Δx.

Therefore $A=\underset{\Delta x\doteq 0}{\mathrm{Lt}}\sum_{a}^{b-\Delta x}f(x)\Delta x=\underset{\Delta x\doteq 0}{\mathrm{Lt}}\sum_{a+\Delta x}^{b}f(x)\Delta x,$

by reasoning the same as before.*

25. Let us apply the general reasoning to a concrete example. Take the curve $y=\frac{1}{10}x^2$, say, and let it be required to find the area between the ordinates where $x=3$ and $x=6$. Use coördinate paper, drawing in the curve very nicely with an irregular draughting curve. The points are located most expeditiously by means of a table of squares. Adhering to our notation, suppose we divide the base, $(b-a)$, into three parts; then $y_1=.9$, $y_2=1.6$, $y_3=2.5$, $y_4=3.6$, $\Delta x\equiv h=1$, $a=3$, $b=6$, $n=3$.

$$\therefore \quad \sum_{1}^{3}y_xh=5.0<A<\sum_{2}^{4}y_xh=7.7.$$

* This method of quadratures, as explained in the last three articles, might be termed the *modern* method of *exhaustions*.

Hence the area, A, differs from 7.7 or from 5.0 by less than $(y_4 - y_1)h = 2.7$ units; also seen from $7.7 - 5.0 = 2.7$.

Next divide the base into six equal parts; then $n = 6$, $\Delta x \equiv h = \frac{1}{2}$, and

$$\begin{aligned} y_1 &= .900 \\ y_2 &= 1.225 \\ y_3 &= 1.600 \\ y_4 &= 2.025 \\ y_5 &= 2.500 \\ y_6 &= \underline{3.025} \\ \sum_1^6 y_x &= 11.275 \end{aligned}$$

$$\therefore \sum_1^6 = 5.6375 \text{ and } \sum_2^7 = \sum_1^6 - y_1 h + y_7 h = 6.9875.$$

Hence $5.6375 < A < 6.9875$, and the error is less than 1.35 units, a closer approximation than before.

Again, take $\Delta x \equiv h = .1$. Then from a table of squares we readily find the numerical values of the consecutive ordinates beginning with the second to be as follows:

$\sum_2^{11} y_x$	$\sum_{12}^{21} y_x$	$\sum_{22}^{31} y_x$
.961	1.681	2.601
1.024	1.764	2.704
1.089	1.849	2.809
1.156	1.936	2.916
1.225	2.025	3.025
1.296	2.116	3.136
1.369	2.209	3.249
1.444	2.304	3.364
1.521	2.401	3.481
1.600	2.500	3.600
12.685	20.785	30.885

$$\therefore \qquad \sum_{2}^{31} = \sum_{2}^{11} + \sum_{12}^{21} + \sum_{22}^{31} = 6.4355$$

and
$$\sum_{1}^{30} = \sum_{2}^{31} - y_{31}h + y_1 h = 6.1655.$$

Hence, $6.4355 > A > 6.1655$, and the error made in taking either number for A is less than 0.27 units; shown also by $(y_{31} - y_1)h = 0.27$.

Of course it is easy to see now, from $(y_{n+1} - y_1)h$, that for 300 subdivisions of $(b-a)$ the error would be less than 0.027; for 3000 subdivisions less than 0.0027; and so on. Let us see if the limit can be found.

26. Introducing the literal values given in the example, we have

$$\sum_{x=a}^{b-\Delta x} \tfrac{1}{10} x^2 \Delta x < A < \sum_{a+\Delta x}^{x=b} \tfrac{1}{10} x^2 \Delta x;$$

that is,

$$\frac{\Delta x}{10}\Big[a^2 + (a+h)^2 + \cdots + (a + \overline{n-1}\,h)^2\Big] < A,$$

$$A < \frac{\Delta x}{10}\Big[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2\Big];$$

$$\frac{\Delta x}{10}\Big[na^2 + 2ah\sum_{1}^{n-1} z + h^2 \sum_{1}^{n-1} z^2\Big] < A,$$

$$A < \frac{\Delta x}{10}\Big[na^2 + 2ah\sum_{1}^{n} z + h^2 \sum_{1}^{n} z^2\Big];$$

$$\frac{b-a}{10n}\left[na^2 + 2a\frac{b-a}{n}\frac{n(n-1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{(n-1)n(2n-1)}{6}\right] < A,$$

$$A < \frac{b-a}{10n}\left[na^2 + 2a\frac{b-a}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6}\right].$$

Whence

$$\frac{b-a}{10}\left[a^2 + a(b-a)\left(1 - \frac{1}{n}\right) + (b-a)^2 \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}\right] < A,$$

$$\frac{b-a}{10}\left[a^2 + a(b-a)\left(1 + \frac{1}{n}\right) + (b-a)^2 \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right] > A.$$

The first members of these last two inequalities are the areas of the R and L systems of rectangles in terms of the *number of rectangles.* Whence, making n equal to 3, 6, 30, and $a=3$, $b=6$, we derive in order,

$$7.7 > A > 5.0,$$

$$6.9875 > A > 5.6375,$$

$$6.4355 > A > 6.1655\,;$$

which agree with the results just calculated in another way. Taking n larger, R becomes larger and L smaller; by taking n large enough, both R and L may be made to differ as little as you please from

$$\tfrac{1}{10}\left[a^2(b-a)+a(b-a)^2+\tfrac{1}{3}(b-a)^3\right] \equiv \frac{b^3}{30}-\frac{a^3}{30}.$$

Therefore the area required cannot differ from $\left(\dfrac{b^3}{30}-\dfrac{a^3}{30}\right)=6.3$, for the data of the problem. It should be noticed that, since no restriction was placed upon a and b, the formula $\left(\dfrac{b^3}{30}-\dfrac{a^3}{30}\right)$ is a general expression for the area enclosed by the curve $\frac{1}{10}\,x^2$, the axis of x and *any* two ordinates whose abscissæ are a and b. The result of the investigation is indicated by

$$A = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a}^{b-\Delta x} \frac{1}{10} x^2 \Delta x = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a+\Delta x}^{b} \frac{1}{10} x^2 \Delta x = \frac{b^3}{30}-\frac{a^3}{30}.$$

27. The Symbol $\int$.* We indicated $\underset{\Delta x \doteq 0}{\mathrm{Lt}} \dfrac{\Delta y}{\Delta x}$ by $\dfrac{dy}{dx}$; then let us indicate

$$\underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a \pm \Delta x}^{b \pm \Delta x} f(x)\Delta x \text{ by } \int_a^b f(x)\,dx.$$

Hence we have the equation

$$A = \int_a^b \frac{x^2}{10}\,dx = \frac{b^3}{30}-\frac{a^3}{30}.$$

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* This symbol, the long s, was introduced by Leibnitz, and stands for "*limit of* sum."

28. It was shown in the preceding part of this chapter how the solution of the problem of *areas* can be made to depend upon the solution of the problem of a *limit of a sum.* Many of the most important problems in *mechanics* can be made to depend upon a limit of a sum. We will take a simple example. Suppose we wish to find the space described in a given time by a freely falling body. We know by experiment that the velocity gains a definite number of units in each unit of time. This gain per second in velocity is called the *acceleration* of the body, and is represented by g. Near the surface of the earth $g = 32.2$, in the foot-pound-second system of units; this meaning that the velocity increases 32.2 feet per second in a second.

Solution. Let v_0 be the velocity at the beginning of the given time-interval, and measure time from that instant, representing it by θ. Divide the total interval, t, into n equal parts, each $\frac{t}{n} \equiv \Delta\theta$, and represent the elapsed time at the beginning of the rth partial interval by $\theta_r \equiv \overline{r-1}\,\Delta\theta$. Then the velocities at the beginning and end of the rth interval are $v_0 + g\theta_r$, $v_0 + g\theta_{r+1}$.

Whence $\sum\limits_{r=1}^{r=n}(v_0 + g\theta_r)\Delta\theta < s < \sum\limits_{r=2}^{r=n+1}(v_0 + g\theta_r)\Delta\theta$; where s is the required actual space passed over during the time t.

But
$$\sum_{r=2}^{n+1}(v_0 + g\theta_r)\Delta\theta - \sum_{r=1}^{n}(v_0 + g\theta_r)\Delta\theta$$
$$= g\left[\sum_{2}^{n+1} - \sum_{1}^{n}\right]\theta_r\Delta\theta$$
$$= g(\theta_{n+1} - \theta_1)\Delta\theta = gt\Delta\theta$$

vanishes when $\Delta\theta = 0$, from which it follows that

$$s = \operatorname*{Lt}_{\Delta\theta \doteq 0}\sum_{r=1}^{n}(v_0 + g\theta_r)\Delta\theta = \operatorname*{Lt}_{\Delta\theta \doteq 0}\sum_{r=1}^{n} v_0\Delta\theta + \operatorname*{Lt}_{\Delta\theta \doteq 0}\sum_{r=1}^{n} g\theta_r\Delta\theta.$$

Now
$$\operatorname*{Lt}_{\Delta\theta \doteq 0}\sum_{r=1}^{n} v_0\Delta\theta = \operatorname*{Lt}_{\Delta\theta \doteq 0} v_0\sum \Delta\theta = v_0 t,$$

and, resorting to the method of summation in Chapter I., using the identity

$$x\Delta x \equiv \frac{(x+\Delta x)^2}{2} - \frac{x^2}{2} - \frac{\Delta x^2}{2},$$

$$\underset{\Delta\theta \doteq 0}{\mathrm{Lt}} \sum_{r=1}^{n} \theta_r \Delta\theta = \underset{\Delta\theta \doteq 0}{\mathrm{Lt}} \sum_{\theta=0}^{t-\Delta\theta} \theta\Delta\theta$$

$$= \underset{\Delta\theta \doteq 0}{\mathrm{Lt}} \left\{ \frac{1}{2}\left[\sum_{\Delta\theta}^{t} - \sum_{0}^{t-\Delta\theta} \right] \theta^2 - \frac{1}{2} \sum_{r=1}^{n} \Delta\theta^2 \right\}$$

$$= \underset{\Delta\theta \doteq 0}{\mathrm{Lt}} \left\{ \frac{t^2}{2} - \frac{0^2}{2} - \frac{n\Delta\theta^2}{2} \right\}$$

$$= \underset{\Delta\theta \doteq 0}{\mathrm{Lt}} \left\{ \frac{t^2}{2} - \frac{t\Delta\theta}{2} \right\} = \frac{t^2}{2}.$$

Therefore $$s = \int_0^t (v_0 + g\theta)\,d\theta = v_0 t + g\frac{t^2}{2},$$

the well-known formula for the space passed over by a freely falling body with initial velocity v_0.

29. Attraction. We know that the sun, the planets, and other celestial bodies attract one another; and the attraction of two leaden balls has been approximately measured.

Newton's Law of Universal Gravitation may be stated as follows: *Any two material particles in the universe attract each other with a force which varies as the product of their masses directly and the square of their distance apart inversely.*

For analytical purposes we may explain that *material particle* means *indefinitely small body*, and that *distance apart* refers to a distance indefinitely great as compared to the linear dimensions of the particles. So that all parts of each particle are supposed to be at the same distance from the other particle; or, to put it still more mathematically, we are to suppose it possible to accumulate attracting matter at a point.

30. Problem. Required the attraction exerted by a very thin homogeneous straight rod, whose mass per unit length is δ, upon a very small mass m' in the produced axis of the rod, distant a units from the near end and b from the remote end.

SOLUTION. Let the line $\overline{MN}$ represent the rod, and m' the small mass in its axis, taken so that $\overline{Mm'} \equiv a$ and $\overline{Nm'} \equiv b$. Consider a portion of the rod Δx in length, and let x be the distance of m' from the nearer end of Δx. Then $\delta\Delta x$ is the mass of the portion, and $k\dfrac{m'\delta\Delta x}{x^2}$ would be the attraction between Δx and m' if all the mass, $\delta\Delta x$, were concentrated at the distance x from m'. Thus, if a_x be the attraction of the masses considered, we shall have

$$k\frac{m'\delta\Delta x}{x^2} > a_x > k\frac{m'\delta\Delta x}{(x+\Delta x)^2}. \quad k = \text{a constant.}$$

$$\therefore \quad k\delta m' \sum_{x=a}^{b-\Delta x}\frac{\Delta x}{x^2} > A > k\delta m' \sum_{a+\Delta x}^{b}\frac{\Delta x}{x^2},$$

where A is the required attraction. From this, by taking the difference between the extreme members of the double inequality, we easily show that $A = km'\delta\displaystyle\int_a^b \frac{dx}{x^2}$. It remains to find $\displaystyle\int_a^b \frac{dx}{x^2}$.

From the identity $\dfrac{\Delta x}{x^2} \equiv \dfrac{1}{x} - \dfrac{1}{x+h} + \dfrac{h^2}{x^2(x+h)}$, we have

$$\sum_a^{b-\Delta x}\frac{\Delta x}{x^2} = \left[\sum_a^{b-\Delta x} - \sum_{a+\Delta x}^{b}\right]\frac{1}{x} + \sum_a^{b-h}\frac{h^2}{x^2(x+h)}$$

$$= \frac{1}{a} - \frac{1}{b} + h^2\sum_a^{b-h}\frac{1}{x^2(x+h)}$$

$$= \frac{1}{a} - \frac{1}{b} + h^2 n\rho,$$

where ρ is a finite quantity, the arithmetical mean of the n terms of $\displaystyle\sum_a^{b-h}\frac{1}{x^2(x+h)}$; a quantity which lies between $\dfrac{1}{a^2(a+h)}$ and $\dfrac{1}{(b-h)^2b}$, and consequently, for any admissible value of $\Delta x \equiv h$, between the limits $\dfrac{1}{a^3}$ and $\dfrac{1}{b^3}$. Also $nh = n\Delta x = (b-a)$.

$$\therefore \quad A = km'\delta\int_a^b\frac{dx}{x^2} = km'\delta \operatorname*{Lt}_{h \doteq 0}\left\{\frac{1}{a} - \frac{1}{b} + \rho(b-a)h\right\} = km'\delta\left(\frac{1}{a} - \frac{1}{b}\right).$$

31. From the foregoing examples we gain some idea of the utility and importance of the notion of a limit of a sum. The following problems will illustrate further:

EXERCISES

1. Let the student solve the example of the area of the curve $\frac{1}{10}x^2$ by finding the sums in terms of Δx instead of n, using an identity which he should be able to form for himself.

2. Solve the example of the falling body by the method of the example of area. In this form it involves an arithmetical progression.

3. $\underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a+\Delta x}^{b} x^3 \Delta x = ?$ Use $x^3 \Delta x \equiv \frac{(x+h)^4}{4} - \frac{x^4}{4} - \frac{(6x^2+4xh+h^2)h^2}{4}$,

where $\Delta x \equiv h$, letting ρ be a quantity which lies between the greatest and least of the n terms of $\sum \frac{6x^2+4xh+h^2}{4}$.

Ans. $\int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4}$.

4. Find $\int_a^b \frac{dx}{x^3}$.

SOLUTION. $$\int_a^b \frac{dx}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a+\Delta x}^{b} \frac{\Delta x}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a}^{b-\Delta x} \frac{\Delta x}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a \pm S\Delta x}^{b \pm T\Delta x} \frac{\Delta x}{x^3},$$

where T and S are any finite numbers; thus $\int_a^b \frac{dx}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a}^{b} \frac{\Delta x}{x^3}$.

Also $$\frac{\Delta x}{x^3} \equiv \frac{1}{2x^2} - \frac{1}{2(x+\Delta x)^2} + \frac{(3x+2\Delta x)\Delta x^2}{2x^3(x+\Delta x)^2}.$$

$$\therefore \int_a^b \frac{dx}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_{a}^{b} \frac{\Delta x}{x^3} = \underset{\Delta x \doteq 0}{\mathrm{Lt}} \left\{ \frac{1}{2a^2} - \frac{1}{2(b+\Delta x)^2} + \Delta x^2 n\rho \right\},$$

where ρ and $\Delta x n\rho = (b + \Delta x - a)\rho$ are finite quantities, n being the number of terms in $\sum \frac{(3x+2\Delta x)\Delta x^2}{2x^3(x+\Delta x)^2}$. Thus $\Delta x^2 n\rho$ vanishes when $\Delta x = 0$.

Therefore $$\int_a^b \frac{dx}{x^3} = \frac{1}{2a^2} - \frac{1}{2b^2}.$$

5. $\int_a^b \cos x dx = ?$

Use $\cos x\Delta x \equiv \left\{\sin\left(x+\frac{h}{2}\right) - \sin\left(x-\frac{h}{2}\right)\right\}\dfrac{\frac{h}{2}}{\sin\frac{h}{2}}.$

Ans. $\sin b - \sin a$.

6. $\underset{\Delta x \doteq 0}{\text{Lt}} \sum_a^b \frac{\Delta x}{x} = ?$

SOLUTION. We know from algebra that $\log(1+y) \equiv y - \frac{y^2}{2} + \frac{y^3}{3} - R$, where, if $y < 1$, $R < \frac{y^3}{3} < \frac{y^2}{2}$.

Hence,

$$\log\left(1+\frac{\Delta x}{x}\right) + \frac{1}{2}\left(\frac{\Delta x}{x}\right)^2 > \frac{\Delta x}{x} > \log\frac{x+\Delta x}{x} + \frac{1}{2}\left(\frac{\Delta x}{x}\right)^2 - \frac{1}{3}\left(\frac{\Delta x}{x}\right)^3,$$

and

$$\sum_a^b \log + \frac{\Delta x}{2}\sum_a^b \frac{\Delta x}{x^2} > \sum_a^b \frac{\Delta x}{x} > \sum_a^b \log + \frac{\Delta x}{2}\sum \frac{\Delta x}{x^2} - \frac{\Delta x^2}{3}\sum \frac{\Delta x}{x^3}.$$

But $$\sum_a^b \log = \log\frac{b+\Delta x}{a},\quad \text{Lt}\sum\frac{\Delta x}{x^2} = \frac{1}{a} - \frac{1}{b},$$

$$\text{Lt}\sum\frac{\Delta x}{x^3} = \frac{1}{2a^2} - \frac{1}{2b^2},\quad \text{Lt}\frac{\Delta x}{2} = 0 \text{ and } \text{Lt}\frac{\Delta x^2}{3} = 0.$$

Therefore $$\int_a^b \frac{dx}{x} = \log\frac{b}{a}.$$

7. Let the student see if he can name the most important difficulty which arises in the solution of these problems.

8. Explain the meaning of the equation

$$\int_a^b f(x)\,dx = \underset{\Delta x \doteq 0}{\text{Lt}}\left\{\sum_a^{s+t\Delta x} f(x)\Delta x + \sum_s^b f(x)\Delta x\right\},$$

and tell why it is that any finite number of vanishing terms may be omitted from the summation without affecting the limit. Illustrate graphically.

CHAPTER IV

OBSERVATIONS UPON THE INTEGRAL CALCULUS

32. BEFORE taking up this chapter the student should have acquainted himself with the rules of differentiation and the forms of ordinary derivatives. He may have observed in the problems of summation, that to find the limits of sums of the form $\sum \phi(x)\Delta x$, it was necessary, in the general case, to have an identity of the form

$$\phi(x)\Delta x \equiv \psi(x) - \psi(x+\Delta x) + F(x, \Delta x)\Delta x^2.$$

The fundamental theorem of the integral calculus puts into mathematical language a rule for finding the limit of any sum, of the kind considered, provided an identity of the right form can be found; and the rules and formulæ of the integral calculus afford a method for the discovery of the essential form of the identity when it exists.

33. **Fundamental Theorem.** $\int_a^b f'(x)dx = f(b) - f(a)$. Or, more explicitly,

$$\underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_a^b \psi'(x)\Delta x = \psi(b) - \psi(a),$$

where $\psi'(x)$ is any function of x and $\psi(x)$ any function whose differential coefficient with respect to x is $\psi'(x)$.

PROOF. We know that

$$\underset{h \doteq 0}{\mathrm{Lt}} \frac{\phi(x+h) - \phi(x)}{h} = \phi'(x);$$

hence

$$\frac{\phi(x+h) - \phi(x)}{h} + \epsilon \equiv \phi'(x),$$

where ϵ is a quantity vanishing with $h \equiv \Delta x$.

Thus $$\phi'(x)\Delta x \equiv \phi(x+h) - \phi(x) + h\epsilon$$

and $$\underset{\Delta x \doteq 0}{\mathrm{Lt}} \sum_a^b \phi'(x)\Delta x \equiv \underset{h \doteq 0}{\mathrm{Lt}} \{\phi(b+h) - \phi(a) + h\Sigma\epsilon\}$$

$$= \phi(b) - \phi(a) + \underset{h \doteq 0}{\mathrm{Lt}}\, hn\alpha$$

$$= \phi(b) - \phi(a) + \underset{h \doteq 0}{\mathrm{Lt}}\, (b+h-a)\alpha$$

$$= \phi(b) - \phi(a)\,;$$

where n is the number of terms in the summation and α the arithmetical mean of the n terms of the type ϵ, a quantity vanishing with h. Therefore the theorem

$$\int_a^b f'(x)dx = f(b) - f(a). \qquad \text{Q.E.D.}$$

34. The importance and application of the theorem will be shown immediately by using it in the solution of the problems already solved.

In finding the area of the parabola $\frac{1}{10}x^2$, it was shown that the area $= \int_a^b \frac{x^2}{10}dx$. But $\frac{x^3}{30}$ is a function whose derivative is $\frac{x^2}{10}$. Therefore, by the theorem,

$$A = \int_a^b \frac{x^2}{10}\,dx = \frac{b^3}{30} - \frac{a^3}{30}.$$

In the case of the falling body the space passed over was shown to be $\int_0^t (v_0 + g\theta)d\theta$. But $v_0\theta + g\frac{\theta^2}{2}$ is a function whose derivative is $v_0 + g\theta$. Therefore

$$s = v_0 t + g\frac{t^2}{2}.$$

In the next problem the attraction was $km'\delta\int_a^b \frac{dx}{x^2}$; and $-\frac{1}{x}$ is a function whose derivative is $\frac{dx}{x^2}$. Therefore

$$A = km'\delta\left\{\left(-\frac{1}{b}\right) - \left(-\frac{1}{a}\right)\right\} = km'\delta\left(\frac{1}{a} - \frac{1}{b}\right).$$

The remaining problems are tabulated in order in the three columns below, the first giving the problem, the second the function whose derivative is to be of a given form, and the third the result:

$\int_a^b x^3 dx,$	$\frac{x^4}{4},$	$\frac{b^4}{4} - \frac{a^4}{4}.$
$\int_a^b \frac{dx}{x^3},$	$-\frac{1}{2x^2},$	$\frac{1}{2a^2} - \frac{1}{2b^2}.$
$\int_a^b \cos x dx,$	$\sin x,$	$\sin b - \sin a.$
$\int_a^b \frac{dx}{x},$	$\log x,$	$\log b - \log a.$

35. The expression $\int_a^b \phi(x)dx$ is called the *definite integral of $\phi(x)$, or of $\phi(x)dx$, between the limits a and b.* Now the fundamental theorem tells us that the value of this definite integral can be found by finding the function whose differential coefficient is $\phi(x)$, substituting first b, and then a, for x in this function, and subtracting the latter result from the former. We then very properly define $\int \phi(x)dx$, to mean any function of x, which, when differentiated, gives $\phi(x)$. Thus, let $\psi(x)$ be such a function; then $\frac{d}{dx}\left[\psi(x) + c\right] = \frac{d}{dx}\psi(x) = \phi(x)$. Hence there are an infinite number of such functions, any two of which differ by a constant, but any one of which satisfies the conditions of the theorem. We call $\int \phi(x)dx$ the *indefinite* integral of $\phi(x)$; it is a function which, when found, generally leads to the definite integral between the given limits. The indefinite integrals in the above problems are $\frac{x^3}{30} + c$, $v_0\theta + \frac{g}{2}\theta^2 + c$, $-\frac{1}{x} + c$, $\frac{x^4}{4} + c$, $-\frac{1}{2x^2} + c$, $\sin x + c$, and $\log x + c$. Hence we see that $\int_a^b F(x)dx$ is not a function of x. What is it a function of? Is $\int F(x)dx$ a function of x?

36. The indefinite integral does not always exist for every form of function taken for derivative, and no general rule for finding it can be given without the use of transcendental functions. Thus the processes of the integral calculus are indirect, and depend upon our knowledge of the forms of derivatives gained in the differential calculus. We mean that the integral calculus depends upon the differential calculus, but that *definite* integrals, and consequently *indefinite* integrals, may be found by the ordinary processes of algebra without reference to differentiation; if this be called *integration*, then integration does not necessarily depend upon differentiation, a point very important to understand.

37. After reading the integral calculus, the student should follow it up with references to the *Encyclopædia Britannica*, at "Infinitesimal Calculus" (Historical Introduction) and at "Archimedes"; Cajori's *History of Mathematics*, at "Hippias of Elis," "Antiphon," "Eudoxus," and elsewhere. Thus it was that the germ of the infinitesimal analysis had already assumed tangible form even in the remotest of antiquity.

SEVEN LESSONS

IN THEORY OF

INVERSIONS OF ORDER

AND

DETERMINANTS

BY

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PREFACE

No student should enter upon the study of analytic geometry without an elementary knowledge of that part of the theory of determinants which treats of *elimination* and the *solution of simultaneous equations*. This will be the more readily conceded when it is shown that such knowledge may be gained in fewer than a dozen lessons. Seven introductory lessons in theory of inversions and determinants would scarcely suffice *per se* to make a lasting impression upon the student; but if the elementary principles thus learned are immediately applied to the solution of problems in analytic geometry the good gained from such a brief course should be very considerable. In whatever manner the student be introduced to a subject, it is, in most cases, only by constant recurrence and many varied illustrations that the value of a process is finally realized.

It is thought by the writer that the treatment of inversions as applied to determinants is new in some respects, and attention is invited to the *numbered* theorems and related matter which form a chain of reasoning leading up to Laplace's development.

University of Minnesota
April, 1902.

INVERSIONS AND DETERMINANTS

CHAPTER I

INVERSIONS OF ORDER

1. If several consecutive symbols of a sequence, as letters of the alphabet, or numerals, be written in a line in any order, then an inversion of natural order, or simply an **inversion,** occurs whenever any symbol follows another which it should *naturally* precede. In *bac*, 5647, 4321, *JKIHG*, 564213, there are respectively 1, 2, 6, 9, 12 inversions.

2. In considering any such line of symbols we shall number their positions to the right beginning at the left. Then the **order of position** of any symbol in the line is of the 1st, 2nd, 3d, degree according as it occupies the 1st, 2nd, 3d, position; and so on. The **order of a symbol,** and the degree of the order, is taken with reference to the natural sequence of the symbols in the line: it is the same as the order of the position it would occupy if the symbols were arranged in natural order in the positions of the line. It will be frequently convenient to refer to a symbol, or position, as being *even* or *odd,* meaning that the *order* of the symbol, or position, is of even or odd degree.

3. **Theorem I.** *In any line of symbols capable of arrangement in natural sequence, if any two symbols be interchanged the number of inversions in the line will be increased or decreased by an odd number.*

The interchange of any two symbols occupying consecutive

positions obviously causes a loss or gain of a single inversion. If there are m symbols intervening between the two to be interchanged, then the transfer of each of the two to the position of the other obviously causes a loss or gain of one inversion with each intermediate; moreover there is also a loss or gain of one inversion between the two symbols interchanged. Therefore there are $2m+1$, an odd number, of losses and gains *together*. But the change in the number of inversions is the *difference* between losses and gains, and if the sum of two integers be odd their difference is odd. Therefore the change in the number of inversions is odd in any case.

4. A **complex symbol** can be formed by uniting into a single symbol of two simple parts any two symbols chosen from as many different sets of sequences of the kind we have been considering. A *triply* complex symbol may be formed by choosing from three sets of sequences; and so on. As illustrations, (11), $\overset{h}{\underset{x}{\cdot}}$, b_3, A_4, Q^a, a^2, B^c, are doubly, and $A\overset{b}{\underset{y}{-}}$, $m^{1/3}$, ${}^3X_a$, triply complex. We shall confine our remarks to the first kind.

5. Definition. The **order of a complex symbol** is the sum of the orders of its simple parts.

6. **Theorem II.** *If to any number of consecutive letters taken in any order, the same number of consecutive numerical suffixes be attached, one suffix to each letter, then upon writing these complex symbols in line in any order at pleasure, the total number of inversions among both letters and suffixes will be either always odd or always even.*

In any *one* interchange of two symbols the number of inversions among either letters or suffixes is changed by an odd number (Theorem I); and the sum or difference of two odd numbers is an even number. Hence the total number of inversions after one interchange remains either odd or even as it was. But by successive interchanges two at a time the symbols can be brought into any prescribed order one at a time. The truth of the theorem is apparent.

Example. The number of inversions in $a_1b_2c_3d_4$, $d_4c_3a_1b_2$, $d_4c_3b_2a_1$, or any other permutation of the group, is respectively 0, 10, 12, even.

Scholium. The greatest possible number of inversions in such groups is $n(n-1)$, where n is the number of complex symbols.

7. **Theorem III.** *In any line of doubly complex symbols, whose letters and suffixes are the members of corresponding sequences, the total number of inversions due to the presence of any specified complex symbol, H_x, is even or odd according as the order of that symbol is even or odd.*

If the order of H_x is even, then H and x are both even or both odd; consequently there must be present an *even* number of letters and suffixes together which are of lower orders than H, x respectively. If H_x is odd, then H, x are one odd the other even, and there must be present an *odd* number of letters and suffixes together of lower orders than H, x respectively. Therefore the number of inversions due to H_x in the *first* position of the line is even or odd according as H_x is even or odd, since the only inversions due to H_x in that position are with letters and suffixes of lower orders than H, x respectively. But the number of inversions due to H_x is always even or always odd, independent of the arrangement of the line, since in any one interchange, and consequently in any succession of interchanges, it is impossible for H_x to gain or lose an odd number of inversions with any other doubly complex symbol. The theorem follows.

Cor. *The number of inversions due to the symbol H_x is even or odd according as $(-1)^r$ is $+$ or $-$; r being the order of H_x.*

Cor. *If there be a line of simple symbols arranged in any order, then the number of inversions in the line due to the presence of any specified symbol of the sequence will be even or odd according as the sum of the orders of the specified symbol and its position is even or odd.*

For a line of doubly complex symbols may be written with its letters in natural order and its order of suffixes corresponding to the order of symbols in the given line. Then the order of position of any suffix is the same as the order of its literal partner, and the number of inversions in the given line is the

same as the number in the written line. The corollary follows from this and Theorem III.

8. Definition. The **order of a group,** of two or more complex symbols, is the sum of the orders of the constituents of the group.

9. **Theorem IV.** *The number of inversions in any line of doubly complex symbols, due to the presence of any specified two or more such symbols, is even or odd according as* $(-1)^{s+i}$ *is* $+$ *or* $-$; *s being the order of the specified group and i its number of inversions.*

Let S be the sum of the numbers of inversions respectively due to each symbol of the specified group considered separately. Then S includes each inversion occuring among the specified symbols *twice,* and each inversion between a symbol within the group and another without the group but once. Therefore $S-i$ is the number of inversions due to the specified group in the line. But S and s are both even or both odd (Theorem III and properties of numbers). Therefore the number of inversions due to the group is even or odd according as $S-i$, and consequently as $s-i$, is even or odd; that is according as $(-1)^{s-i}$, and therefore as $(-1)^{s+i}$, is $+$ or $-$.

Cor. *Theorem III is a special case of Theorem IV, since then* $s=r$ *and* $i=0$.

Examples.

1. Is the number of inversions in $a_2\, c_1\, f_7\, g_6\, d_3\, e_4\, b_5$, due to the presence of $a_2\, g_6\, d_3\, b_5$, even or odd? For ease of enumeration represent the groups by $\frac{1\ 3\ 6\ 7\ 4\ 5\ 2}{2\ 1\ 7\ 6\ 3\ 4\ 5}$ and $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$. In the latter $(s-i)=30-5=25$, and $(-1)^{25}$ is $-$. Hence there should be an *odd* number of inversions due to $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$; and by actual count there are 17 in $\frac{1\ 3\ 6\ 7\ 4\ 5\ 2}{2\ 1\ 7\ 6\ 3\ 4\ 5}$ and 2 in $\frac{3\ 6\ 5}{1\ 7\ 4}$, leaving 15, an *odd* number, due to $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$.

2. In Ex. 1 $s-i=24$ for $\frac{3\ 6\ 5}{1\ 7\ 4}$; and accordingly there are 12 inversions due to $\frac{3\ 6\ 5}{1\ 7\ 4}$.

3. In $d_1\, a_2\, c_4\, b_5\, e_3$ there are 6 inversions due to the presence of $c_4\, e_3\, d_1$ for which $s-i=16$ an *even* number.

CHAPTER II

ARRAYS AND DETERMINANTS

10. The expression $\begin{vmatrix} 12 & 4 & -71 & 6 \\ 51 & 32 & -16 & 31 \\ -77 & 22 & 4 & 2 \\ 9 & 1 & -8 & 17 \end{vmatrix}$ exhibits $4^2=16$ quantities included between the verticals. In general an exhibit of n^2 quantities may be formed in **square array** by arranging them thus in n vertical columns and n horizontal rows. Three exhibits in square array of the 9 quantities 1, 2, 3,...9, are

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}, \quad \begin{vmatrix} 1 & 5 & 9 \\ 2 & 6 & 7 \\ 3 & 4 & 8 \end{vmatrix}.$$

11. In any square array we will number the columns in consecutive order to the right beginning at the left, and the rows in consecutive order downward beginning at the top. Thus, the third column, or column 3, in the first array above contains the quantities —71, —16, 4, —8, while row 2 contains 51, 32, —16, 31. Instead of numbering the columns and rows 1, 2, 3,...., it will frequently be convenient to *letter* them a, b, c,... ; i.e. column 3≡column c, etc.

12. With two sequences of n simple symbols each n^2 different complex symbols can be formed by writing every symbol of one sequence n times and affixing in any uniform manner, to each one so written, the n symbols of the other seqence, singly, in succession. In illustration, the $3^2=9$ symbols a_1, a_2, a_3, $b_1, b_2, b_3, c_1, c_2, c_3$, are composed of the three letters a, b, c, and the three figures 1, 2, 3. Again, the nine complex symbols $\overset{1}{\dot{1}}, \overset{2}{\dot{1}}, \overset{3}{\dot{1}}, \overset{1}{\dot{2}}, \overset{2}{\dot{2}}, \overset{3}{\dot{2}}, \overset{1}{\dot{3}}, \overset{2}{\dot{3}}, \overset{3}{\dot{3}}$, or the nine (11), (12), (13), (21), (22), (23), (31), (32), (33), are formed from 1, 2, 3, and 1, 2, 3.

Hence we may *represent* any n^2 quantities in square array by any one of

$$\begin{vmatrix} a_1 & a_2 & a_3 \ldots a_n \\ b_1 & b_2 & b_3 \ldots b_n \\ c_1 & c_2 & c_3 \ldots c_n \\ \ldots & \ldots & \ldots \\ m_1 & m_2 & m_3 \ldots m_n \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \ldots m_1 \\ a_2 & b_2 & c_2 \ldots m_2 \\ a_3 & b_3 & c_3 \ldots m_3 \\ \ldots & \ldots & \ldots \\ a_n & b_n & c_n \ldots m_n \end{vmatrix}, \quad \begin{vmatrix} (11) & (12) & (13) \ldots (1n) \\ (21) & (22) & (23) \ldots (2n) \\ (31) & (32) & (33) \ldots (3n) \\ \ldots & \ldots & \ldots \\ (n1) & (n2) & (n3) \ldots (nn) \end{vmatrix}^{*}.$$

$$\begin{vmatrix} \genfrac{}{}{0pt}{}{1}{1} & \genfrac{}{}{0pt}{}{2}{1} & \genfrac{}{}{0pt}{}{3}{1} \ldots \genfrac{}{}{0pt}{}{n}{1} \\ \genfrac{}{}{0pt}{}{1}{2} & \genfrac{}{}{0pt}{}{2}{2} & \genfrac{}{}{0pt}{}{3}{2} \ldots \genfrac{}{}{0pt}{}{n}{2} \\ \genfrac{}{}{0pt}{}{1}{3} & \genfrac{}{}{0pt}{}{2}{3} & \genfrac{}{}{0pt}{}{3}{3} \ldots \genfrac{}{}{0pt}{}{n}{3} \\ \ldots & \ldots & \ldots \\ \genfrac{}{}{0pt}{}{1}{n} & \genfrac{}{}{0pt}{}{2}{n} & \genfrac{}{}{0pt}{}{3}{n} \ldots \genfrac{}{}{0pt}{}{n}{n} \end{vmatrix}^{*}.$$

13. In the first array just written the order of any letter is the same as the order of the row containing it, and the order of any suffix is the same as the order of the column containing it. In the second array letters correspond to columns and suffixes to rows. The modes of arrangement in the remaining two arrays are apparent. It is evidently immaterial which one of these, or other similar forms, is used, simply for the purpose of *representing* a square array.

14. DEFINITIONS. The n^2 quantities of an array are called **constituents,** or **elements,** of the array. The **order of a constituent** with respect to rows or columns is respectively the same as the order of the row or column containing it. The order of a constituent with respect to rows *and* columns is the sum of the orders of the row and column containing it. A square array of n rows and n columns, and consequently containing n^2 constituents, is said to be of the nth order. Care must be taken to distinguish different classes of order.

15. Hence we may not only represent an array of quantities by a similar array of complex symbols, but each symbol

---

* Here of course the figures, and the numbers resulting from their combination, do not have their usual signification; but any symbol, as $\genfrac{}{}{0pt}{}{2}{3}$ or (32), represents the *quantity* which stands in the *same position* in the array represented.

may be taken to represent the *value* and *position* of the corresponding quantity of the array represented, by having the orders of every pair of associated simple components correspond to the two orders of position referred respectively to rows and columns as has been explained.

16. **Coordination of products.** The product of any n constituents of a square array of the nth order, so selected that one and only one constituent is taken from each row and one and only one from each column, we shall refer to as a *product of the array.* There are other ways of coordinating products, but this is the only one with which we shall be concerned.

17. The diagonal row of constituents through the top left hand corner of the array is called the **principal diagonal,** and the product of the constituents of this diagonal is the **principal product** of the array.

18. In considering any such product, take the factors in order to the right, beginning at the left. Then, corresponding to this *order of factors,* there is an order in which they were taken from columns and an order in which from rows. We shall speak of such an order as the *order of choice, or order of selection,* from columns or rows as the case may be; when such choice has been made in the natural order of rows or columns we shall refer to it as the *natural* order of choice.

Illustration. What is the order of choice from rows and columns in the product, $pble$, of the array $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{vmatrix}$?

The order of selection from columns is 3241, or $CBDA$; the order of choice from rows is 4132, or $DACB$; the order of choice from columns *and* rows is represented by $c_4b_1d_3a_2$, or by $\dfrac{3\ 2\ 4\ 1}{4\ 1\ 3\ 2}$.

19. **Theorem.** *In any product of an array, coordinated as in Art. 16, the total number of inversions of natural order in the orders*

of choice from rows and columns is always even or always odd irrespective of the order of factors.

The general square array may be represented by

$$\left|\begin{array}{cccccc} \overset{1}{\underset{1}{\cdot}} & \overset{1}{\underset{2}{\cdot}} & \overset{1}{\underset{3}{\cdot}} & \cdots\cdots & \overset{1}{\underset{n}{\cdot}} \\ \overset{2}{\underset{1}{\cdot}} & \overset{2}{\underset{2}{\cdot}} & \overset{2}{\underset{3}{\cdot}} & \cdots\cdots & \overset{2}{\underset{n}{\cdot}} \\ \overset{3}{\underset{1}{\cdot}} & \overset{3}{\underset{2}{\cdot}} & \overset{3}{\underset{3}{\cdot}} & \cdots\cdots & \overset{3}{\underset{n}{\cdot}} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \overset{n}{\underset{1}{\cdot}} & \overset{n}{\underset{2}{\cdot}} & \overset{n}{\underset{3}{\cdot}} & \cdots\cdots & \overset{n}{\underset{n}{\cdot}} \end{array}\right|;$$

the natural orders of superior and inferior components corresponding respectively to natural orders of rows and columns. Take any product of the array, as $\alpha\,\beta\,\gamma \ldots . \sigma$, represented by $\overset{h}{\underset{s}{\cdot}}\,\overset{p}{\underset{t}{\cdot}}\,\overset{x}{\underset{y}{\cdot}} \cdot \cdot \overset{r}{\underset{v}{\cdot}}$; then the orders of arrangement of superior and inferior components, corresponding to the order of factors, is the same as the orders of choice from rows and columns respectively. But (Theorem II) the total number of inversions among the complex symbols in the product is always even or always odd irrespective of the order of factors. Therefore, the total number of inversions of natural order, in the order of choice from rows and columns, is always even or always odd irrespective of the order of factors.

Cor. *If the sign-factor* $(-1)^i$ *be attached to any product of the kind mentioned, i being the total number of inversions of natural order in the orders of choice from rows and columns, then the sign with which the product is to be ultimately affected is always —, or always —, irrespective of the order of factors.*

20. The **determinant** of a square array of the nth order is the algebraic sum of all the possible products of the n^2 constituents taken n together, limited by the condition that one and only one constituent is taken from each row and one and only one from each column; the sign with which any product is ultimately affected being $+$, or $-$, according as there is an even or odd total number of inversions of natural order in the orders of choice from rows and columns.

21. The determinant of a square array is frequently represented by $\triangle$. Hence the definition of a determinant, expressed in mathematical symbols, is $\triangle \equiv \Sigma\,(-1)^i\,(a_1 b_2 c_3 \ldots . m_n)$;

where n is the degree of the array and i the total number of inversions in the term to which it refers; the summation to extend over all the possible terms that can be brought into coordination by the rule of Art. 16.

22. The square array $\begin{vmatrix} a_1 & b_1 \ldots m_1 \\ a_2 & b_2 \ldots m_2 \\ \ldots & \ldots \\ a_n & b_n \ldots m_n \end{vmatrix}$ will be represented by $|a_1 b_2 \ldots m_n|_c$, it being understood that $a_1, b_2, \ldots m_n$, are but the constituents of the principal diagonal of the array. The suffix c denotes that the superior components of the complex symbols correspond to columns. In like manner

$$\begin{vmatrix} a_1 & a_2 \ldots a_n \\ b_1 & b_2 \ldots b_n \\ \ldots & \ldots \\ m_1 & m_2 \ldots m_n \end{vmatrix}$$

will be represented by $|a_1 b_2 \ldots m_n|_r$, r referring to rows in the same manner that c does to columns.

23. When the terms of a determinant have been written out with their proper signs, or when the process of writing them out according to any definite rule has been represented, the determinant is said to be *developed*.

CHAPTER III

FORMATION OF A DETERMINANT

24. In the sequel an array will be frequently spoken of as being a determinant, meaning however the determinant of the array; just as in algebra x frequently means *value of* x. In the same way it will be convenient to refer to rows and columns of a determinant, meaning rows and columns of an array from which the determinant may be developed. When no distinction is necessary the word *line* will be used indifferently for *row* or *column*.

25. There are n! terms of $\triangle$. Let $r_1, r_2, \ldots . r_n$, represent the n rows and $c_1, c_2, \ldots . c_n$ the n columns of a determinant, $\triangle$, of the nth order. By $(r_h, c_k) \equiv {h \atop k}$, we shall mean the constituent at the junction of the hth row and kth column. Associating any r, as r_h, with any c, as c_k, we shall have, ${h \atop k}$, a constituent of $\triangle$. Associating any one of the remaining r's, as r_k, with any one of the remaining c's, as c_h, another constituent, ${k \atop l}$, is represented which is not in the row or column of ${h \atop k}$. Again, associating any r still remaining, r_j, with any c still remaining, c_t, we have ${j \atop t}$, a constituent not in any row or column already chosen. Continuing this process until all the rows and columns have been exhausted, we have a set of constituents coordinated by the rule of Art. 16. The product of these constituents together with the sign-factor $(-1)^i$, i being the number of inversions in ${h \atop k}{k \atop h}{j \atop t} \cdots {p \atop q}$, is, by definition, a term of $\triangle$: it is clear that any permutation of $r_1, r_2 \ldots r_n$ combined with any permutation of $c_1, c_2 \ldots . c_n$, one r with one c, will lead to a term of $\triangle$. But there are $n!$ permutations of n symbols taken all at a time. Hence there are $(n!)^2$ ways of writing terms of $\triangle$, and no more. But each term may be written in $n!$ ways. Therefore there are $(n!)^2 \div n! = n!$ terms of $\triangle$, and no more.

26. It is important to bear in mind that such expressions as ${h \atop k}{k \atop h}{j \atop t} \cdots {p \atop q}$, where the superior components refer to rows and

the inferior to columns, or vice versa, fully *indicate* the orders of choice from rows and columns and *represent* the constituents of a product.

Examples.

1. What are the signs of the terms $i\,b\,q\,g$, $l\,e\,c\,n$, $f\,k\,m\,d$, of the determinant of
$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{vmatrix}?$$

The following scheme shows the method of determining signs:

	$i\,b\,q\,g$	$l\,e\,c\,n$	$f\,k\,m\,d$
term:	$i\ b\ q\ g$	$l\ e\ c\ n$	$f\ k\ m\ d$
order of selection, rows:	3 1 4 2	$\gamma\ \beta\ \alpha\ \delta$	2 3 4 1
order of selection, columns:	1 2 4 3	$d\ a\ c\ b$	2 3 1 4
number inversions:	4	7	5
sign of the term:	+	−	−

Scholium. The factors of the terms may be rearranged in natural order as to rows or columns. Then the process may be as under:

$i\ b\ g\ q$	$c\ e\ l\ n$	$d\ f\ k\ m$
3 1 2 4	$C\ A\ D\ B$	$\delta\ \beta\ \gamma\ \alpha$
2	3	5

Of course the same signs result as before.

2. Is $e\,p\,j\,c$ a term of $\triangle$ in Ex. 1?

27. Let $|a_1\ b_2 \ldots . m_n|$ represent the determinant of a square array of the nth order (Art. 22) to be developed. Then selecting constituents from columns in natural order for every term, permuting into new order the order of choice from rows for each new term, is represented by writing $a\,b \ldots . m$ in natural order $n!$ times and affixing the n suffixes each time in a new one of their $n!$ permutations. Thus there will be $n!$ *different* terms, which are therefore all the terms of $\triangle$, and the sign of any term will be the sign of $(-1)^i$, i being the number of inversions among the suffixes of that term; there being no inversions among letters. The algebraic sum of these terms is $\triangle$. This is merely a systematic method of doing what was outlined in Art. 25.

If we select from *rows* in natural order, permuting the order of choice from columns, we shall evidently arrive at the same result, but the sign is determined by the number of in-

versions among the letters, there being no inversions among suffixes.

28. Let the determinant be represented by $|a_1 b_2 \ldots m_n|_r$. Then the scheme of development with respect to natural order of *rows* will be identically the same, *with regard to symbolic constituents*, as it was in the development of $|a_1 b_2 \ldots m_n|_c$ with respect to natural order of *columns*; but only the symbols of the principal diagonal represent the same constituents of the determinant in the two cases. Either scheme, however, must lead to the development of the determinant represented since both accord with the definition of a determinant.

Examples.

Develop $\begin{vmatrix} 2 & -6 & 4 \\ 5 & -3 & 9 \\ -7 & 1 & -8 \end{vmatrix}$. Represent it by $|a_1 b_2 c_3|_c$.

Then letters correspond to columns and suffixes to rows. Hence selecting from columns in natural order and permuting the orders of choice from rows is indicated by writing $a_1b_2c_3$, $a_1b_3c_2$, $a_2b_1c_3$, $a_2b_3c_1$, $a_3b_1c_2$, $a_3b_2c_1$; the letters being written in natural order and the suffixes permuted in all possible orders, each permutation corresponding to a term of Δ. Counting the inversions among suffixes in each term, attaching corresponding signs and substituting the values of the constituents, we obtain the 3! terms of Δ; thus

$$\begin{aligned}\Sigma(-1)^i(a_1b_2c_3) &= +(2)(-3)(-8)-(2)(1)(9)-(5)(-6)(-8)\\ &\quad +(5)(1)(4)+(-7)(-6)(9)-(-7)(-3)(4)\\ &= 48-18-240+20+378-84\\ &= 104\end{aligned}$$

Scholium. To follow the symbolic definition (Art. 21) to the letter we should write

$\Delta = (-1)^0(a_1b_2c_3) + (-1)^1(a_1b_3c_2) + (-1)^1(a_2b_1c_3) + (-1)^2(a_2b_3c_1) + (-1)^2(a_3b_1c_2) + (-1)^3(a_3b_2c_1)$; but this is unnecessary as the number of inversions in any term is +, or −, according as the number of inversions in that term is even or odd.

2. Represent Δ in Ex. 1. by $|a_1b_2c_3|_r$ and develop. Compare results. Compare Art. 28.

3. Let the order of choice from columns in Ex. 1. be *b c a* for every term. Compare results.

Suggestion. $i \equiv$ number of inversions among both letters and suffixes.

4. Develop $\begin{vmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 5 & 3 & 2 \\ 2 & 4 & 1 & 4 \end{vmatrix}$. Ans. 55.

CHAPTER IV

PROPERTIES OF DETERMINANTS

29. **Theorem.** *A determinant is unaltered by changing its rows into corresponding columns and its columns into corresponding rows.*

Represent Δ by $| \, a_1, b_2, \ldots . m_n \, |_c$ (Art. 27). Change columns to corresponding rows in the manner of the theorem and represent the resulting determinant by Δ_1. Then $\Delta_1 \equiv | \, a_1, b_2, \ldots . m_n \, |_r$. But (Art. 28) $| \, a_1, b_2, \ldots . m_n \, |_c \equiv | \, a_1, b_2, \ldots . m_n \, |_r$. Therefore $\Delta = \Delta_1$. Q. E. D.

Scholium. Here of course we do not mean the arrays are identical but their determinants are.

30. **Theorem.** *A determinant is unaltered in absolute value, but changed in sign, by the interchange of any two rows or any two columns.*

Let $\Delta_1 \equiv | \, a_1 b_2 . . h_s . . k_t . . m_n \, |_c$ be the determinant whose columns corresponding to h, k, are to be interchanged. Then $\Delta_2 \equiv | \, a_1 b_2 . . k_s . . h_t . . m_n \, |_c$ is the resulting determinant. Developing both with respect to *natural order of columns*, any term of Δ_1, as $(-1)^i \, a_v \, b_w . . h_p . . k_q . . m_u$, will appear in Δ_2 as $(-1)^j \, a_v \, b_w . . k_q . . h_p . . m_u$, where the order of suffixes is the same except that p and q are interchanged. Therefore (Theorem I.) the signs of the two terms are different. But this reasoning applies to any term of Δ_1. Therefore for every term of Δ_1 there is another in Δ_2 of equal absolute value but of contrary sign. Therefore $\Delta_1 = - \Delta_2$.

Let the student give the proof for the case in which two rows are interchanged.

31. **Theorem.** *The determinant of a square array, in which two rows or two columns are identical, is equal to zero.*

Any interchange in the manner of the theorem should change the sign of Δ. But the *value* of Δ is evidently iden-

Articles 29, 30, 31, 32, 42 43, 44, 33, in this order, are to be arranged as CHAPTER V, PROPERTIES OF DETERMINANTS, following MINOR DETERMINANTS as CHAPTER IV.

tically the same in the two cases. Thus Δ is unaltered in value by changing its sign. Therefore $\Delta = 0$.

32. **Theorem.** *If all the constituents of one line be multiplied by the same factor, the determinant itself will be multiplied by that factor.*

Let $|a_1 b_2 \dots m_n|_c = \Delta$. Multiply any line as the kth row, or pth column, by m. Then all the constituents of the form x_k, or all of the form p_y, become mx_k or mp_y respectively. But every term of Δ contains one and only one constituent of each of the forms mentioned. Therefore every term of Δ, and consequently Δ itself, is multiplied by m in either case.

Cor. From the above and Art. 31 it is concluded that *if two rows or two columns differ only by a constant factor the determinant must vanish.*

33. The following examples are solved by applying one or more of the foregoing theorems, as will be evident upon examination of the reductions:

Exercises.

1. $$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 1 & 7 \\ 3 & 1 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 0$$

The last step is to subtract three times the first column from the third. Both steps might have been done in one reduction.

2. $$\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 3 & 9 & 6 \end{vmatrix} = 6\begin{vmatrix} 7 & 11 & 2 \\ 13 & 15 & 5 \\ 1 & 3 & 1 \end{vmatrix} = 6\begin{vmatrix} 5 & 5 & 2 \\ 8 & 0 & 5 \\ 0 & 0 & 1 \end{vmatrix} = -240$$

The last determinant develops easily as there is only one term not vanishing.

3. Develop $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

They reduce to $\begin{vmatrix} 1 & 4 & 7 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 0$.

4. $\begin{vmatrix} 7 & 14 & 14 \\ 12 & 9 & 24 \\ 8 & 7 & 16 \end{vmatrix} = 0$, because the third column is twice the first.

5. $\begin{vmatrix} 4 & 1 & 1 \\ 2 & -2 & 1 \\ -2 & -8 & 1 \end{vmatrix} \equiv \begin{vmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \\ -2 & -8 & 1 \end{vmatrix} = 0$; since in the last determinant the 2nd row is twice the first.

6. $\begin{vmatrix} 0 & 3 & 1 \\ 9 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} \equiv \begin{vmatrix} 0 & 9 & 1 \\ 3 & 0 & 1 \\ 1 & 6 & 1 \end{vmatrix} \equiv \begin{vmatrix} 0 & 9 & 1 \\ 3 & 9 & 1 \\ 1 & 9 & 1 \end{vmatrix} = 0.$

The first reduction is made by dividing the 1st column by 3 and multiplying the 2nd by 3.

7. $\begin{vmatrix} 7 & -2 & 5 \\ -3 & 1 & 4 \\ 4 & 6 & -3 \end{vmatrix} = -\begin{vmatrix} 5 & -2 & 7 \\ 4 & 1 & -3 \\ -3 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 1 & -3 \\ 5 & -2 & 7 \\ -3 & 6 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & -3 \\ 12 & -2 & 7 \\ 1 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 14 & -2 & 1 \\ -5 & 6 & 22 \end{vmatrix} = -313.$$

CHAPTER V

MINOR DETERMINANTS

34. When any number of rows and the same number of columns of a square array are deleted, the remaining square array is called a **minor determinant.** The square array of constituents which lie at the junctions of deleted rows and columns is also a minor. The two minors thus formed are called **complementary minors.** A *first* minor is formed by striking out one row and one column; a *second* minor by striking out two rows and two columns; and so on. The first minor formed by suppressing the row and column through any particular constituent, as x, is called the minor of that constituent and is represented by Δ_x. A constituent and its first minor are complementary minors.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ are first minors of } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and are respectively Δ_{c_3}, Δ_{b_2}, Δ_{a_1}.

35. It follows from the definition of a minor that the natural order of its rows and columns is also natural order with respect to rows and columns of the original determinant. Therefore, *the number of inversions in any term of the minor referred to natural order of rows and columns of the minor, is the same as the number of inversions referred to rows and columns of the original determinant.*

36. **Theorem V.** *The sum of all the terms of Δ which contain a given constituent, k, is $(-1)^r k \Delta_k$, where r is the order of k referred to rows and columns of Δ.*

Since no term of Δ which contains k, can contain any other constituent from the row or column through k, it follows that every term of Δ which contains k must be the product of $(-1)^r k$ (Theorem III and Art. 35) and some term of Δ_k; con-

versely, the product of $(-1)^r k$ and *any* term, T_y, of Δ_k is obviously a term of Δ. Therefore the sum of all the terms of Δ which contain k is $\Sigma\,(-1)^r k\,T_y \equiv (-1)^r k\,\Sigma\,T_y = (-1)^r k\,\Delta_k$.

Cor. *A determinant of the nth order may be written in the form* $\Sigma\,k_y K_y$, *where* $k_1, k_2, \ldots . k_n$, *are all the constituents in any line of* Δ *and* $K_y \equiv (-1)^r\,\Delta_{k_y}$.

For $(-1)^r\,k_y\,\Delta_{k_y}$ is the sum of all the terms of Δ containing k_y; hence $\Sigma(-1)^r k_y\,\Delta_{k_y}$ is the sum of *all* the terms which contain an element from the k line of Δ. But every term of Δ must, by definition, contain one element from the k line; therefore there are no terms of Δ besides those written.

37. The factors $K_1, K_2, \ldots . K_n$, are the **cofactors** of $k_1, k_2, \ldots . k_n$ respectively; they are the minors $\Delta_{k_1}, \Delta_{k_2}, \ldots . \Delta_{k_n}$, with positive or negative sign according as the orders of the complementary minors $k_1, k_2, \ldots . k_n$, with respect to rows and columns of Δ, are even or odd. When a determinant is expressed in the form $\Sigma\,k_y K_y$ it is said to be developed with reference to the line containing the elements k.

Scholium. It follows that $(-1)^\sigma \Delta_k$ is the cofactor of k, if σ is the sum of the orders of the constituents of any term of Δ_k with reference to rows and columns of Δ.

Scholium. The cofactor of a constituent, a_x, will be represented by the capital letter corresponding, as A_x.

Exercises.

1. Find the sum of all the terms of $\begin{vmatrix} 24 & -1 & 14 & 3 \\ 17 & -4 & 9 & 12 \\ -3 & 71 & 13 & 21 \\ 6 & -2 & 3 & 6 \end{vmatrix}$ which contain 13; also of those which contain 9.

By the preceding theorem and Cor., Art. 32, we obtain, after expanding the last, $13\begin{vmatrix} 24 & -1 & 13 \\ 17 & -4 & 12 \\ 6 & -2 & 6 \end{vmatrix} = 0$ and $-9\begin{vmatrix} 24 & -1 & 3 \\ -3 & 7 & 21 \\ 6 & -2 & 6 \end{vmatrix}$ $= 15876$.

2. Develop $\begin{vmatrix} \alpha & \lambda & \kappa \\ \theta & \mu & \psi \\ \beta & \phi & \omega \end{vmatrix}$ with respect to the 2nd column.

Ans. $\Delta = -\lambda\begin{vmatrix} \theta & \psi \\ \beta & \omega \end{vmatrix} + \mu\begin{vmatrix} \alpha & \kappa \\ \beta & \omega \end{vmatrix} - \phi\begin{vmatrix} \alpha & \kappa \\ \theta & \psi \end{vmatrix}$.

3. $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv c_1 C_1 + c_2 C_2 + c_3 C_3 + c_4 C_4$$

$$\equiv c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}$$

$$+ c_3 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_4 & b_4 & d_4 \end{vmatrix} - c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

4. What is the cofactor of 9, Ex. 1?

5. $$\begin{vmatrix} 3 & 6 & 1 & 3 \\ -7 & -2 & 7 & -5 \\ 4 & -3 & 9 & -6 \\ 5 & 8 & 4 & 2 \end{vmatrix} = -414.$$

38. **Theorem VI.** *The sum of all the terms of Δ such that each term of the sum contains one term of any given minor, Δ_1, is $(-1)^s\Delta_1\Delta_2$; Δ_2 being the complementary minor and s the sum of the orders of the constituents of any term of Δ_1 with reference to rows and columns of Δ.*

Since no term of Δ which contains a given term $(-1)^i\times(hk\ldots.q)$, of Δ_1, can contain any other constituent from any row or column containing $h, k, \ldots. q$, it follows that every term of Δ which contains $(hk\ldots.q)$ must be the product of $(-1)^{s+i}(hk\ldots.q)$ and some term of Δ_2 (Theorem IV. Art. 35. Art. 20.); conversely, the product of $(-1)^{s+i}(hk\ldots.q)$ and any term T of Δ_2 is obviously a term of Δ. Hence the sum of all the terms of Δ which contain $(hk\ldots.q)$ is $(-1)^{s+i}(hk\ldots.q)\Delta_2$. But this reasoning applies to every term of Δ_1; $s+i$ corresponding. Therefore the sum of all the terms of Δ which satisfy the conditions of the theorem is

$$\Sigma(-1)^{s+i}(hk\ldots.q)\,\Delta_2 = (-1)^s\Delta_2\,\Sigma(-1)^i(hk\ldots.q)$$
$$= (-1)^s\Delta_2\,\Delta_1;$$

since s is the *same* for all terms of Δ_1.

Scholium. Since the orders of any two complementary minors are necessarily both even or both odd, s may refer to either Δ_1 or Δ_2 indifferently. The generality of the theorem also includes this.

Cor. *A determinant, Δ, of the nth order, may be written in the form $\Sigma(-1)^s X_m Z_r$; where any term of the sum consists of the product of the sign-factor $(-1)^s$, a minor, as X_m, selected from a given m rows (columns) of Δ, and the complementary minor Z_r, necessarily from the remaining r rows (columns); s being the order of the principal diagonal of either X_m or Z_r as explained above, and no two minors of the form X_m being selected from the same combination of columns (rows).*

For, every term of the $\frac{n_m}{m!}$* terms of $\Sigma(-1)^s X_m Z_r$ obviously contains $m!r!$ terms of Δ, no two of which are alike (Theorem VI). Further; no two terms of Σ, as $(-1)^s X_m Z_r$ and $(-1)^{s'} X_m' Z_r'$, can contain a term common to both, since X_m and X_m' must differ in respect to the constituents of one line at least, and one constituent from that line must appear in every term of $(-1)^s X_m Z_r$ and $(-1)^{s'} X_m' Z_r'$. Therefore $\Sigma(-1)^s X_m Z_r$ is the sum of $m!\,r!\frac{n_m}{m!} = n_{n-r} r! = n!$ terms of Δ, no two of which are alike. But this is all the terms of Δ. The corollary follows.

39. **Problem.** Develop $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$ in the form $\Sigma(-1)^s (a_x b_y)(c_u d_v e_w)$; $(a_x b_y)$ being a minor from the first two columns.

Solution. Selecting the ten minors formed by the association of the first two columns with each of the $5 \cdot 4 \div 2!$ combinations of the five rows two at a time, we obtain $(a_1 b_2)$, $(a_1 b_3)$, $(a_1 b_4)$, $(a_1 b_5)$, $(a_2 b_3)$, $(a_2 b_4)$, $(a_2 b_5)$, $(a_3 b_4)$, $(a_3 b_5)$, $(a_4 b_5)$; where $(a_1 b_2) \equiv |a_1 b_2|_c$, etc., the suffix c being dropped since it is understood that letters correspond to columns. Hence, enumerating orders, attaching sign factors, multiplying by complementary minors and summing, there results $\Delta = (-1)^6 (a_1 b_2)(c_3 d_4 e_5)$ $+ (-1)^7 (a_1 b_3)(c_2 d_4 e_5) + (-1)^8 (a_1 b_4)(c_2 d_3 e_5) + (-1)^9 (a_1 b_5)(c_2 d_3 e_4)$ $+ (-1)^8 (a_2 b_3)(c_1 d_4 e_5) + (-1)^9 (a_2 b_4)(c_1 d_3 e_5) + (-1)^{10} (a_2 b_5)(c_1 d_3 e_4)$ $+ (-1)^{10} (a_3 b_4)(c_1 d_2 e_5) + (-1)^{11} (a_3 b_5)(c_1 d_2 e_4)$ $+ (-1)^{12} (a_4 b_5)(c_1 d_2 e_3)$;
the mode of development clearly being to write the 10 combinations of the five suffixes, taken two together, in natural order in the places of x, y, in $(a_x b_y)$, for the ten minors from the first two columns of Δ.

*$n_m \equiv n(n-1)(n-2)\ldots(n-m+1)$ is the meaning of the notation employed. By the conditions of the theorem we have $n = m + r$.

40. Develop $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ in the form $(-1)^s\,(x_2y_4)\,(u_1v_3)$; (x_2y_4) being a minor from the 2nd and 4th rows. We must have

$$\Delta = (-1)^9\,(a_2b_4)\,(c_1d_3) + (-1)^{10}\,(a_2c_4)\,(b_1d_3) + (-1)^{11}\,(a_2d_4)\,(b_1c_3) + (-1)^{11}\,(b_2c_4)\,(a_1d_3) + (-1)^{12}\,(b_2d_4)\,(a_1c_3) + (-1)^{13}\,(c_2d_4)\,(a_1b_3)$$

Scholium. To find the order of x_2y_4 simply add 6 to the sum of the orders of the letters in every case.

41. The development of a determinant of the third order by the method of Art. 37, and the development of a determinant of the fourth order by Arts. 38, 39 and 40, should be thoroughly learned by the student. It is not necessary to introduce the sign-factor as the proper signs may be attached directly.

CHAPTER VI

SIMULTANEOUS EQUATIONS

42. **Theorem.** *If the elements of one column of a determinant be multiplied in order by the cofactors of the corresponding elements of any other column, then the sum of the products will be zero.*

Take a determinant of the 4th order,

$$\begin{vmatrix} m_1 & n_1 & r_1 & s_1 \\ m_2 & n_2 & r_2 & s_2 \\ m_3 & n_3 & r_3 & s_3 \\ m_4 & n_4 & r_4 & s_4 \end{vmatrix}:$$

multiply the elements of the second row by the cofactors of the fourth and take the sum;

$$\text{thus:}\quad X = m_2 M_4 + n_2 N_4 + r_2 R_4 + s_2 S_4.$$
$$\text{But}\quad \triangle = m_4 M_4 + n_4 N_4 + r_4 R_4 + s_4 S_4.$$

Whence X may be found by changing m_4 to m_2, n_4 to n_2, and so on, in $\triangle$. It follows that $X = 0$.

The reasoning applies to columns as well as to rows and to any determinant.

43. **Theorem.** *If each element of a column of a determinant is the sum of two quantities, the determinant can be expressed as the sum of two determinants of the same order.*

$$\text{Thus;}\quad \begin{vmatrix} a_1 & b_1+\beta_1 & c_1 \\ a_2 & b_2-\beta_2 & c_2 \\ a_3 & b_3+\beta_3 & c_3 \end{vmatrix} \equiv (b_1+\beta_1)B_1 + (b_2+\beta_2)B_2 + (b_3+\beta_3)B_3$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

44. **Theorem.** *A determinant is unaltered in value by adding to all the elements of any row the same multiples of the corresponding elements of another row.*

Thus: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1+ma_2+na_3 & a_2 & a_3 \\ b_1+mb_2+nb_3 & b_2 & b_3 \\ c_1+mc_2+nc_3 & c_2 & c_3 \end{vmatrix}$; for the second member $\equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & a_2 & a_3 \\ mb_2 & b_2 & b_3 \\ mc_2 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} na_3 & a_2 & a_3 \\ nb_3 & b_2 & b_3 \\ nc_3 & c_2 & c_3 \end{vmatrix}$; and the last two determinants vanish.

45. Let it be required to solve the three simultaneous equations

$$\begin{aligned} a_1 x+b_1 y+c_1 z &= m_1, \\ a_2 x+b_2 y+c_2 z &= m_2, \\ a_3 x+b_3 y+c_3 z &= m_3. \end{aligned}$$

First. To find x. Multiply the first equation through by A_1, the second by A_2, the third by A_3 and add. The coefficients of x, y, z in the resulting equation are respectively

$$\begin{aligned} a_1 A_1+a_2 A_2+ a_3 A_3 &= \Delta, \\ b_1 A_1+ b_2 A_2+ b_3 A_3 &= 0, \\ c_1 A_1+ c_2 A_2+ c_3 A_3 &= 0; \end{aligned}$$

where Δ is the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ formed of the array of coefficients taken as they appear in the equations. Therefore $\Delta x = m_1 A_1+m_2 A_2+m_3 A_3 = (m_1 b_2 c_3)$ and $x = (m_1 b_2 c_3) \div \Delta$.

Then multiplying through by B_1, B_2, B_3, instead of A_1, A_2, A_3, the value of y is found to be $(a_1 m_2 c_3) \div \Delta$. Finally, the value of z is found in similar manner to be $(a_1 b_2 m_3) \div \Delta$.

Examples.

1. Find the values of w, x, y, z, in

$$
\begin{aligned}
a_1 w + a_2 x + a_3 y + a_4 z &= a \\
b_1 w + b_2 x + b_3 y + b_4 z &= b \\
c_1 w + c_2 x + c_3 y + c_4 z &= c \\
d_1 w + d_2 x + d_3 y + d_4 z &= d
\end{aligned}
$$

Ans. $w = \dfrac{(a\ b_2\ c_3\ d_4)}{(a_1\ b_2\ c_3\ d_4)}$, $x = \dfrac{(a_1\ b\ c_3\ d_4)}{(a_1\ b_2\ c_3\ d_4)}$, $y = \dfrac{(a_1\ b_2\ c\ d_4)}{\Delta}$,

$$z = \frac{(a_1\ b_2\ c_3\ d)}{\Delta.}.$$

2. Solve

$$
\begin{aligned}
x + 2y + 3z &= 6 \\
2x + 3y + 4z &= 3 \\
3x + y + 5z &+ 1.
\end{aligned}
$$

Solution.

$$x = \frac{(m_1\ b_2\ c_3)}{\Delta} = \frac{\begin{vmatrix} 6 & 2 & 3 \\ 3 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 1 & 5 \end{vmatrix}} = \frac{\begin{vmatrix} 4 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{vmatrix}} = \frac{4\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix}} = \frac{44}{-6} = -\frac{22}{3};$$

$$y = \frac{(a_1\ m_2\ c_3)}{\Delta} = \frac{\begin{vmatrix} 1 & 6 & 3 \\ 2 & 3 & 4 \\ 3 & 1 & 5 \end{vmatrix}}{-6} = \frac{\begin{vmatrix} 1 & 6 & 2 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{vmatrix}}{-6} = \frac{\begin{vmatrix} -1 & 3 & 0 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix}}{-3} = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}{3} = -\frac{1}{3};$$

$$z = \frac{(a_1\ b_2\ m_3)}{\Delta} = \frac{\begin{vmatrix} 1 & 2 & 6 \\ 2 & 3 & 3 \\ 3 & 1 & 1 \end{vmatrix}}{-6} = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 2 & 1 & 0 \\ 3 & -2 & 0 \end{vmatrix}}{-6} = \frac{4\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}{-6} = \frac{14}{3}.$$

Scholium. Remembering the theorems the reductions are very apparent, excepting possibly the second reduction for the denominator of x, where the determinant is found from the preceding denominator by adding the 2nd row to the 3d and subtracting twice the 1st from the sum; then the 1st from the 2nd, the first row remaining unchanged.

Scholium. This method of solving simultaneous equations should be employed throughout analytic geometry in order to become familiar with the process.

46. **Homogeneous Equations.** In the case of $(n-1)$ homogeneous equations containing n unknowns we cannot determine the unknowns, but their ratios may be expressed in the manner of the following:

Let
$$a_1 x+b_1 y+c_1 z = 0,$$
$$a_2 x+b_2 y+c_2 z = 0,$$

be two homogeneous equations in three unknowns. Add below another equation of the same general form,

$$a_3 x+b_3 y+c_3 z = \lambda,$$

a_3, b_3, c_3, λ, being undetermined, and represent the array of coefficients by $\triangle$.

$$\therefore x = \frac{\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ \lambda & b_3 & c_3 \end{vmatrix}}{\triangle} = \frac{\lambda A_3}{\triangle},\ y = \frac{\lambda B_3}{\triangle},\ z = \frac{\lambda C_3}{\triangle}.$$

$$\therefore \frac{x}{A_3} = \frac{y}{B_3} = \frac{z}{C_3}.$$

The method is easily extended to any number of equations.

Example.

$$2w+3x-4y+5z = 0,$$
$$4w-x+12y-2z = 0,$$
$$6w-7x-20y+z = 0.$$

$$\therefore A_4 = -376,\ B_4 = -752,\ C_4 = 188,\ D_4 = 752.$$

$$\therefore \frac{w}{2} = \frac{x}{4} = \frac{y}{-1} = \frac{z}{-4}.$$

CHAPTER VII

ELIMINANTS AND DISCRIMINANTS

47. **Elimination.** To eliminate the *two* quantities x, y, from the *three* equations

$$a_1 x+b_1 y+c_1 = 0, \qquad (1)$$
$$a_2 x+b_2 y+c_2 = 0, \qquad (2)$$
$$a_3 x+b_3 y+c_3 = 0. \qquad (3)$$

Multiply (1), (2), (3), in order by C_1, C_2, C_3, and add. The sum is $c_1 C_1+c_2 C_2+c_3 C_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This expresses the condition that the three equations are simultaneous in x, y; that is *consistent*, or capable of being satisfied by the same set of values of x, y.

Similarly the *four* equations

$$a_1 x+b_1 y+c_1 z+d_1 = 0,$$
$$a_2 x+b_2 y+c_2 z+d_2 = 0,$$
$$a_3 x+b_3 y+c_3 z+d_3 = 0,$$
$$a_4 x+b_4 y+c_4 z+d_4 = 0,$$

between the *three* quantities x, y, z, are consistent if

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

In general $\overline{n-1}$ quantities involved in a system of n simultaneous equations may be eliminated by arranging the terms of each equation in corresponding order in one member and equating to zero the determinant of the array so formed by the coefficients. This determinant may be called the **eliminant**

or **resultant** of the system; the reader is referred, however, to the higher theory of equations for the precise definition of the *resultant*.*

Examples.

Eliminate x and y from the three equations

$$a_1 x+b_1 y+c_1 z+d_1 = 0,$$
$$a_2 x+b_2 y+c_2 z+d_2 = 0,$$
$$a_3 x+b_3 y+c_3 z+d_3 = 0.$$

Ans. $\begin{vmatrix} a_1 & b_1 & c_1z+d_1 \\ a_2 & b_2 & c_2z+d_2 \\ a_3 & b_3 & c_3z+d_3 \end{vmatrix} = 0.$

48. **Sylvester's method** of eliminating x from any two rational, integral equations in x is illustrated by the following examples:

1. Eliminate x from the equations

$$ax^2+bx+c = 0 \text{ and } \alpha x^2+\beta x+\gamma = 0.$$

We derive

$$\begin{aligned} ax^3+bx^2+cx &= 0, \\ ax^2+bx+c &= 0, \\ \alpha x^3+\beta x^2+\gamma x &= 0, \\ \alpha x^2+\beta x+\gamma &= 0. \end{aligned}$$

Whence, treating x^3, x^2 and x as so many unknown quantities we obtain

$$\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ \alpha & \beta & \gamma & 0 \\ 0 & \alpha & \beta & \gamma \end{vmatrix} = 0;$$

an equation free of x.

2. $ax^3+bx^2+cx+d = 0$ and $px^2+qx+r = 0$.

Ans. $\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ p & q & r & 0 & 0 \\ 0 & p & q & r & 0 \\ 0 & 0 & p & q & r \end{vmatrix} = 0$

Scholium. It should be observed that the two given equations cannot be simultaneous unless the coefficients are such as to cause the eliminant to vanish.

* See Burnside and Panton on Theory of Equations, Vol. II, p. 69.

49. **Discriminants.** It is shown in the theory of equations that if $f(x)$ have equal roots $f'(x)$ must have at least one root in common with $f(x)$. It is also shown that $f(x)$ and $f'(x)$ cannot vanish together unless $f(x)$ have equal roots. Therefore the *necessary* and *sufficient* condition that $f(x)$ have equal roots is that the resultant of the equations $f(x)=0, f'(x)=0$, vanish. When a resultant is formed in this way with reference to a function, $f(x)$, and its derivate, $f'(x)$, it is called the *discriminant** of the equation $f(x)=0$. Hence the necessary and sufficient condition that $f(x)$ have equal roots is that its discriminant vanish.

Examples.

1. Form the discriminant of $x^2-2\,a\,x+a^2=0$.

$$2f(x) \equiv 2x^2-4\,a\,x+2\,a^2,\quad x\,f'(x) \equiv 2\,x^2-2\,a\,x;$$
$$\therefore\ 2f(x)-x\,f'(x) \equiv -2\,a\,x+2\,a^2,\quad f'(x) \equiv 2\,x-2\,a;$$

the eliminant of which is the discriminant,

$$\begin{vmatrix} -2a & 2a^2 \\ 2 & -2a \end{vmatrix} \equiv -a\begin{vmatrix} 2 & -2a \\ 2 & -2a \end{vmatrix} = 0,$$

of $x^2-2\,a\,x+a^2$. Hence we infer that $f(x)$ has equal roots as should have been seen at a glance.

2. Form the discriminant of $x^3+x^2-5\,x+3$.

$$x\,f'(x) \equiv 3\,x^3+2\,x^2-5\,x,\quad n\,f(x) \equiv 3\,x^3+3\,x^2-15\,x+9;$$
$$\therefore\ n\,f(x)-x\,f'(x) \equiv x^2-10\,x+9,\quad f'(x) \equiv 3\,x^2+2\,x-5$$

and the discriminant is the eliminant of

$$\begin{aligned} x^3-10\,x^2+9\,x &= 0,\\ x^2-10\,x+9 &= 0,\\ 3\,x^3+2\,x^2-5\,x &= 0,\\ 3\,x^2+2\,x-5 &= 0. \end{aligned}$$

$$\text{Ans.}\quad \begin{vmatrix} 1 & -10 & 9 & 0 \\ 0 & 1 & -10 & 9 \\ 3 & 2 & -5 & 0 \\ 0 & 3 & 2 & -5 \end{vmatrix} = 0.$$

*The reader is again referred to the theory of equations for the exact definitions of *resultant* and *discriminant*.

Scholium. A method for finding the equal roots of $f(x)$ is to equate the H. C. F. of $f(x)$ and $f'(x)$ to zero and solve for the equal roots.

3. The discriminant of $a_0x^3+3a_1x^2+3a_2x+a_3 = 0$ expressed as a determinant is

$$\begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix}.$$

50. We should not confine ourselves to any one method of development but learn to make use of several; thus Ex. 2, p. 23, solves about as easily by Art. 41 as by the theorems made use of. Sometimes, however, one method possesses advantages over another and should be employed.